## Lecture Notes

## Mathematical Ecnomics

Guoqiang TIAN<br>Department of Economics<br>Texas A\&M University<br>College Station, Texas 77843<br>(gtian@tamu.edu)

This version: November, 2023

[^0]
## Contents

1 The Nature of Mathematical Economics ..... 1
1.1 Economics and Mathematical Economics ..... 2
1.2 Advantages of the Mathematical Approach ..... 2
1.3 Methodology of Scientific Economic Analysis ..... 3
2 Economic Models ..... 5
2.1 Ingredients of a Mathematical Model ..... 5
2.2 The Real-Number System ..... 6
2.3 The Concept of Sets ..... 7
2.4 Relations and Functions ..... 9
2.5 Types of Function ..... 11
2.6 Functions of Two or More Independent Variables ..... 13
2.7 Levels of Generality ..... 14
3 Equilibrium Analysis in Economics ..... 17
3.1 The Meaning of Equilibrium ..... 17
3.2 Partial Market Equilibrium - A Linear Model ..... 18
3.3 Partial Market Equilibrium - A Linear Model ..... 18
3.4 Partial Market Equilibrium - A Nonlinear Model ..... 20
3.5 General Market Equilibrium ..... 21
3.6 Equilibrium in National-Income Analysis ..... 25
4 Linear Models and Matrix Algebra ..... 27
4.1 Matrix and Vectors ..... 28
4.2 Matrix Operations ..... 31
4.3 Linear (In)dependence of Vectors ..... 34
4.4 Commutative, Associative, and Distributive Laws ..... 36
4.5 Identity Matrices and Null Matrices ..... 37
4.6 Transposes and Inverses ..... 39
5 Linear Models and Matrix Algebra (Continued) ..... 43
5.1 Conditions for Nonsingularity of a Matrix ..... 43
5.2 Test of Nonsingularity by Use of Determinant ..... 45
5.3 Basic Properties of Determinants ..... 51
5.4 Finding the Inverse Matrix ..... 57
5.5 Cramer's Rule ..... 63
5.6 Application to Market and National-Income Models ..... 68
5.7 Quadratic Forms ..... 71
5.8 Eigenvalues and Eigenvectors ..... 76
5.9 Vector Spaces ..... 80
6 Comparative Statics and the Concept of Derivative ..... 87
6.1 The Nature of Comparative Statics ..... 87
6.2 Rate of Change and the Derivative ..... 88
6.3 The Derivative and the Slope of a Curve ..... 90
6.4 The Concept of Limit ..... 91
6.5 Inequality and Absolute Values ..... 94
6.6 Limit Theorems ..... 96
6.7 Continuity and Differentiability of a Function ..... 97
7 Rules of Differentiation and Their Use in Comparative Statics ..... 101
7.1 Rules of Differentiation for a Function of One Variable ..... 101
7.2 Rules of Differentiation Involving Two or More Functions of the Same Variable ..... 104
7.3 Rules of Differentiation Involving Functions of Different Vari- ables ..... 110
7.4 Integration (The Case of One Variable) ..... 115
7.5 Partial Differentiation ..... 117
7.6 Applications to Comparative-Static Analysis ..... 120
7.7 Note on Jacobian Determinants ..... 123
8 Comparative-Static Analysis of General Functions ..... 127
8.1 Differentials ..... 128
8.2 Total Differentials ..... 132
8.3 Rule of Differentials ..... 133
8.4 Total Derivatives ..... 135
8.5 Implicit Function Theorem ..... 139
8.6 Comparative Statics of General-Function Models ..... 145
8.7 Matrix Derivatives ..... 146
9 Optimization: Maxima and Minima of a Function of One Vari- able ..... 149
9.1 Optimal and Extreme Values ..... 149
9.2 Existence of Extremum for Continuous Functions ..... 151
9.3 First-Derivative Test for Relative Maximum and Minimum ..... 152
9.4 Second and Higher Derivatives ..... 157
9.5 Second-Derivative Test ..... 158
9.6 Taylor Series ..... 161
9.7 Nth-Derivative Test ..... 163
10 Exponential and Logarithmic Functions ..... 165
10.1 Exponential Functions ..... 165
10.2 Logarithmic Functions ..... 166
10.3 Derivatives of Exponential and Logarithmic Functions ..... 168
11 Optimization for a Function of Two or More Variables ..... 171
11.1 The Differential Version of Optimization Conditions ..... 171
11.2 Extreme Values of a Function of Two Variables ..... 172
11.2.1 First-Order Condition ..... 173
11.2.2 Second-Order Partial Derivatives ..... 174
11.3 Objective Functions with More than Two Variables ..... 179
11.4 Second-Order Conditions in Relation to Concavity and Con- vexity ..... 184
11.4.1 Concave and Convex Functions ..... 184
11.4.2 Concavity/Convexity and Global Optimization ..... 190
11.5 Economic Applications ..... 191
12 Optimization with Equality Constraints ..... 195
12.1 Effects of a Constraint ..... 196
12.2 Finding the Stationary Values ..... 197
12.3 Second-Order Conditions ..... 203
12.4 Quasiconcavity and Quasiconvexity ..... 209
12.5 Utility Maximization and Consumer Demand ..... 216
13 Optimization with Inequality Constraints ..... 221
13.1 Non-Linear Programming ..... 221
13.2 Kuhn-Tucker Conditions ..... 223
13.3 Economic Applications ..... 228

## Chapter 1

## The Nature of Mathematical Economics

Economics is a complex discipline that may seem deceptively simple but is actually challenging to learn, comprehend, and excel in. While students possess a keen interest in learning about the economy, instructors need to concentrate on teaching economics that comprises the theories and models that underpin the field. This inherent tension arises from the fact that there is no single, tangible "economy" to study; rather, economics is based on a set of theories and models that should be judged by their usefulness, much like tools, rather than a single fact. This can make it difficult to understand the nature of economics.

Furthermore, what is commonly thought of as "facts" in economics are actually predictive statements integrated into theoretical models that involve a lot of math, particularly in microeconomic theory. Understanding these technical concepts and logical reasoning is crucial for mastering the subject, further adding to the challenge of learning economics.

The aim of these lecture notes is to introduce students to the fundamental aspects of mathematical knowledge and methods, such as matrix
algebra, mathematical analysis, and optimization theory. These essential tools are not only necessary, but also greatly helpful in effectively learning and mastering economics in general, and economic theory in particular.

### 1.1 Economics and Mathematical Economics

Economics is a social science that studies decision-making in the face of limited resources. Specifically, it examines how individuals, such as consumers, households, firms, organizations, and government agencies, make trade-off choices that allocate scarce resources among competing uses.

Mathematical economics is a mathematical approach to economic analysis, in which economists use mathematical symbols to state problems and draw upon known mathematical theorems to aid in reasoning.

Since mathematical economics is merely an approach to economic analysis, it should not and does not differ from the nonmathematical approach to economic analysis in any fundamental way. The difference between these two approaches is that in the former, the assumptions and conclusions are stated in mathematical symbols rather than words and in the equations rather than sentences.

### 1.2 Advantages of the Mathematical Approach

The mathematical approach offers several advantages, such as:
(1) It makes language more precise and clarifies the assumptions. This helps to avoid unnecessary debates that can arise from inaccurate verbal language.
(2) It makes analytical logic more rigorous and clearly states the boundary, applicable scope, and conditions for a conclusion
to hold. Otherwise, the abuse of a theory may occur.
(3) Mathematics can help obtain results that cannot be easily attained through intuition.
(4) It helps to improve and extend existing economic theories.

However, it is noteworthy that a good command of mathematics cannot guarantee being a good economist. It also requires a thorough understanding of the analytical framework, research methods of economics, and a good intuition and insight into real economic environments and issues. Studying economics not only requires an understanding of various economic terms, concepts, and results from the perspective of mathematics (including geometry), but more importantly, when those are given by mathematical language or geometric figures, we need to get to their economic meaning and the underlying profound economic thoughts and ideals. Therefore, we should avoid being confused by mathematical formulas or symbols in the study of economics.

### 1.3 Methodology of Scientific Economic Analysis

Scientific economic analysis is essential for studying and solving complex economic and social problems. However, relying solely on theory or practice is insufficient. The "three dimensions" of theoretical logic, practical knowledge, and historical perspective, and the "six natures" of scientificity, rigor, practicality, pertinency, foresight, and intellectual depth are crucial for conducting comprehensive analysis and producing sound conclusions.

The use of empirical quantitative analysis is necessary to confirm theoretical reasoning and logical deduction. Historical experience and lesson$s$ is also crucial for understanding fundamental laws, principles, human
behavior patterns, and values. However, relying solely on historical experience may lead to outdated ideas and hinder economic and social development. Therefore, a balance between deductive reasoning and empirical verification is necessary.

By applying the "three dimensions and six natures" in reasoning, testing, and verification, decision-making can be both scientific and artistic, ensuring that conclusions or reform measures and plans meet the necessary criteria. This comprehensive approach is crucial for studying and solving major economic and social issues. Indeed, all knowledge is presented as history, all science is exhibited as logic, and all judgment is understood in the sense of statistics.

As such, it is not surprising that mathematics and mathematical statistics/econometrics are used as the basic and most important analytical tool$s$ in every field of economics. Therefore, mastering sufficient mathematical knowledge is necessary if you want to learn economics well, conduct economic research, and become a good economist.

All in all, to become a good economist, you need to have an original, creative, and academic way of critical thinking.

## Chapter 2

## Economic Models

### 2.1 Ingredients of a Mathematical Model

An economic model is a theoretical framework used to understand the behavior of economic agents and the workings of the economy. Although it is not necessary for an economic model to be mathematical, most economic models are mathematical in nature. Typically, a mathematical economic model consists of a set of equations that describe the relationships among variables in the model.

The equations in an economic model are derived from a set of analytical assumptions about the behavior of economic agents and the workings of the economy. By specifying the functional form of these relationships, the equations give mathematical form to the assumptions. Then, by applying mathematical operations to these equations, economists seek to derive a set of conclusions that logically follow from the assumptions.

### 2.2 The Real-Number System

Whole numbers such as $1,2, \cdots$ are called positive numbers; these are the numbers most frequently used in counting. Their negative counterparts $-1,-2,-3, \cdots$ are called negative integers. The number 0 (zero), on the other hand, is neither positive nor negative, and it is in that sense unique. Let us lump all the positive and negative integers and the number zero into a single category, referring to them collectively as the set of all integers.

Integers of course, do not exhaust all the possible numbers, for we have fractions, such as $\frac{2}{3}, \frac{5}{4}$, and $\frac{7}{3}$, which - if placed on a ruler - would fall between the integers. Also, we have negative fractions, such as $-\frac{1}{2}$ and $-\frac{2}{5}$. Together, these make up the set of all fractions.

The common property of all fractional number is that each is expressible as a ratio of two integers; thus fractions qualify for the designation rational numbers (in this usage, rational means ratio-nal). But integers are also rational, because any integer $n$ can be considered as the ratio $n / 1$. The set of all fractions together with the set of all integers from the set of all rational numbers.

Once the notion of rational numbers is used, however, there naturally arises the concept of irrational numbers - numbers that cannot be expressed as raios of a pair of integers. One example is $\sqrt{2}=1.4142 \cdots$. Another is $\pi=3.1415 \cdots$.

Each irrational number, if placed on a ruler, would fall between two rational numbers, so that, just as the fraction fill in the gaps between the integers on a ruler, the irrational number fill in the gaps between rational numbers. The result of this filling-in process is a continuum of numbers, all of which are so-called "real numbers." This continuum constitutes the set of all real numbers, which is often denoted by the symbol $\mathbb{R}$.

### 2.3 The Concept of Sets

A set is simply a collection of distinct objects. The objects may be a group of distinct numbers, or something else. Thus, all students enrolled in a particular economics course can be considered a set, just as the three integers 2,3 , and 4 can form a set. The object in a set are called the elements of the set.

There are two alternative ways of writing a set: by enumeration and by description. If we let $S$ represent the set of three numbers 2,3 and 4 , we write by enumeration of the elements, $S=\{2,3,4\}$. But if we let $I$ denote the set of all positive integers, enumeration becomes difficult, and we may instead describe the elements and write $I=\{x \mid x$ is a positive integer $\}$, which is read as follows: " $I$ is the set of all $x$ such that $x$ is a positive integer." Note that the braces are used enclose the set in both cases. In the descriptive approach, a vertical bar or a colon is always inserted to separate the general symbol for the elements from the description of the elements.

A set with finite number of elements is called a finite set. Set $I$ with an infinite number of elements is an example of an infinite set. Finite sets are always denumerable (or countable), i.e., their elements can be counted one by one in the sequence $1,2,3, \cdots$. Infinite sets may, however, be either denumerable (set $I$ above) or nondenumerable (for example, $J=\{x \mid 2<$ $x<5\}$ ).

Membership in a set is indicated by the symbol $\in$ (a variant of the Greek letter epsilon $\epsilon$ for "element"), which is read: "is an element of."

If two sets $S_{1}$ and $S_{2}$ happen to contain identical elements,

$$
S_{1}=\{1,2, a, b\} \text { and } S_{2}=\{2, b, 1, a\}
$$

then $S_{1}$ and $S_{2}$ are said to be equal $\left(S_{1}=S_{2}\right)$. Note that the order of
appearance of the elements in a set is immaterial.
If we have two sets $T=\{1,2,5,7,9\}$ and $S=\{2,5,9\}$, then $S$ is a subset of $T$, because each element of $S$ is also an element of $T$. A more formal statement of this is: $S$ is a subset of $T$ if and only if $x \in S$ implies $x \in T$. We write $S \subseteq T$ or $T \supseteq S$.

It is possible that two sets happen to be subsets of each other. When this occurs, however, we can be sure that these two sets are equal.

If a set have $n$ elements, a total of $2^{n}$ subsets can be formed from those elements. For example, the subsets of $\{1,2\}$ are: $\varnothing,\{1\},\{2\}$ and $\{1,2\}$.

If two sets have no elements in common at all, the two sets are said to be disjoint.

The union of two sets $A$ and $B$ is a new set containing elements belong to $A$, or to $B$, or to both $A$ and $B$. The union set is symbolized by $A \cup B$ (read: " $A$ union $B$ ").

Example 2.3.1 If $A=\{1,2,3\}, B=\{2,3,4,5\}$, then $A \cup B=\{1,2,3,4,5\}$.

The intersection of two sets $A$ and $B$, on the other hand, is a new set which contains those elements (and only those elements) belonging to both $A$ and $B$. The intersection set is symbolized by $A \cap B$ (read: " $A$ intersection $B^{\prime \prime}$ ).

Example 2.3.2 If $A=\{1,2,3\}, A=\{4,5,6\}$, then $A \cup B=\varnothing$.

In a particular context of discussion, if the only numbers used are the set of the first seven positive integers, we may refer to it as the universal set $U$. Then, with a given set, say $A=\{3,6,7\}$, we can define another set $\bar{A}$ (read: "the complement of $A$ ") as the set that contains all the numbers in the universal set $U$ which are not in the set $A$. That is: $\bar{A}=\{1,2,4,5\}$.

Example 2.3.3 If $U=\{5,6,7,8,9\}, A=\{6,5\}$, then $\bar{A}=\{7,8,9\}$.

## Properties of unions and intersections:

$$
\begin{aligned}
& A \cup(B \cup C)=(A \cup B) \cup C \\
& A \cap(B \cap C)=(A \cap B) \cap C \\
& A \cup(B \cap C)=(A \cup B) \cap(A \cup C) \\
& A \cap(B \cup C)=(A \cap B) \cup(A \cap C)
\end{aligned}
$$

### 2.4 Relations and Functions

An ordered pair $(a, b)$ is a pair of mathematical objects. The order in which the objects appear in the pair is significant: the ordered pair $(a, b)$ is different from the ordered pair $(b, a)$ unless $a=b$. In contrast, a set of two elements is an unordered pair: the unordered pair $\{a, b\}$ equals the unordered pair $\{b, a\}$. Similar concepts apply to a set with more than two elements, ordered triples, quadruples, quintuples, etc., are called ordered sets.

Example 2.4.1 To show the age and weight of each student in a class, we can form ordered pairs $(a, w)$, in which the first element indicates the age (in years) and the second element indicates the weight (in pounds). Then $(19,128)$ and $(128,19)$ would obviously mean different things.

Suppose, from two given sets, $x=\{1,2\}$ and $y=\{3,4\}$, we wish to form all the possible ordered pairs with the first element taken from set $x$ and the second element taken from set $y$. The result will be the set of four ordered pairs $(1,2),(1,4),(2,3)$, and $(2,4)$. This set is called the Cartesian product, or direct product, of the sets $x$ and $y$ and is denoted by $x \times y$ (read " $x$ cross $y$ ").

Extending this idea, we may also define the Cartesian product of three sets $x, y$, and $z$ as follows:

$$
x \times y \times z=\{(a, b, c) \mid a \in x, b \in y, c \in z\}
$$

which is the set of ordered triples.
Example 2.4.2 If the sets $x, y$, and $z$ each consist of all the real numbers, the Cartesian product will correspond to the set of all points in a threedimension space. This may be denoted by $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$, or more simply, $\mathbb{R}^{3}$.

Example 2.4.3 The set $\{(x, y) \mid y=2 x\}$ is a set of ordered pairs including, for example, $(1,2),(0,0)$, and $(-1,-2)$. It constitutes a relation, and its graphical counterpart is the set of points lying on the straight line $y=2 x$.

Example 2.4.4 The set $\{(x, y) \mid y \leq x\}$ is a set of ordered pairs including, for example, $(1,0),(0,0),(1,1)$, and $(1,-4)$. The set corresponds the set of all points lying on below the straight line $y=x$.

As a special case, however, a relation may be such that for each $x$ value there exists only one corresponding $y$ value. The relation in example 2.4.3 is a case in point. In that case, $y$ is said to be a function of $x$, and this is denoted by $y=f(x)$, which is read: " $y$ equals $f$ of $x$." A function is therefore a set of ordered pairs with the property that any $x$ value uniquely determines a $y$ value. It should be clear that a function must be a relation, but a relation may not be a function.

A function is also called a mapping, or transformation; both words denote the action of associating one thing with another. In the statement $y=f(x)$, the functional notation $f$ may thus be interpreted to mean a rule by which the set $x$ is "mapped" ("transformed") into the set $y$. Thus we
may write

$$
f: x \rightarrow y
$$

where the arrow indicates mapping, and the letter $f$ symbolically specifies a rule of mapping.

In the function $y=f(x), x$ is referred to as the argument of the function, and $y$ is called the value of the function. We shall also alternatively refer to $x$ as the independent variable (also known as the exogenous variables) and $y$ as the dependent variable (also known as the endogenous variables). The set of all permissible values that $x$ can take in a given context is known as the domain of the function, which may be a subset of the set of all real numbers. The $y$ value into which an $x$ value is mapped is called the image of that $x$ value. The set of all images is called the range of the function, which is the set of all values that the $y$ variable will take. Thus the domain pertains to the independent variable $x$, and the range has to do with the dependent variable $y$.

### 2.5 Types of Function

A function whose range consists of only one element is called a constant function.

Example 2.5.1 The function $y=f(x)=7$ is a constant function.

The constant function is actually a "degenerated" case of what are known as polynomial functions. A polynomial functions of a single variable has the general form

$$
y=a_{0}+a_{1} x+a_{2} x^{2}+\cdots+a_{n} x^{n}
$$

in which each term contains a coefficient as well as a nonnegative-integer power of the variable $x$.

Depending on the value of the integer $n$ (which specifies the highest power of $x$ ), we have several subclasses of polynomial function:

$$
\begin{array}{ll}
\text { Case of } n=0: y=a_{0} & \text { [constant function] } \\
\text { Case of } n=1: y=a_{0}+a_{1} x & \\
\text { Case of } n=2: y=a_{0}+a_{1} x+a_{2} x^{2} & \text { [quadritic function] } \\
\text { Case of } n=3: y=a_{0}+a_{1} x+a_{2} x^{2}+a_{3} x^{3} & {[\text { cubic function }]}
\end{array}
$$

A function such as

$$
y=\frac{x-1}{x^{2}+2 x+4}
$$

in which $y$ is expressed as a ratio of two polynomials in the variable $x$, is known as a rational function (again, meaning ratio-nal). According to the definition, any polynomial function must itself be a rational function, because it can always be expressed as a ratio to 1 , which is a constant function.

Any function expressed in terms of polynomials and or roots (such as square root) of polynomials is an algebraic function. Accordingly, the function discussed thus far are all algebraic. A function such as $y=$ $\sqrt{x^{2}+1}$ is not rational, yet it is algebraic.

However, exponential functions such as $y=b^{x}$, in which the independent variable appears in the exponent, are nonalgebraic. The closely related logarithmic functions, such as $y=\log _{b} x$, are also nonalgebraic.

## Rules of Exponents:

Rule 1: $x^{m} \times x^{n}=x^{m+n}$
Rule 2: $\frac{x^{m}}{x^{n}}=x^{m-n}(x \neq 0)$

Rule 3: $x^{-n}=\frac{1}{x^{n}}$
Rule 4: $x^{0}=1(x \neq 0)$
Rule 5: $x^{\frac{1}{n}}=\sqrt[n]{x}$
Rule 6: $\left(x^{m}\right)^{n}=x^{m n}$
Rule 7: $x^{m} \times y^{m}=(x y)^{m}$

### 2.6 Functions of Two or More Independent Variables

Thus for far, we have considered only functions of a single independent variable, $y=f(x)$. But the concept of a function can be readily extended to the case of two or more independent variables. Given a function

$$
z=g(x, y)
$$

a given pair of $x$ and $y$ values will uniquely determine a value of the dependent variable $z$. Such a function is exemplified by

$$
z=a x+b y \text { or } z=a_{0}+a_{1} x+a_{2} x^{2}+b_{1} y+b_{2} y^{2}
$$

Functions of more than one variables can be classified into various types, too. For instance, a function of the form

$$
y=a_{1} x_{1}+a_{2} x_{2}+\cdots+a_{n} x_{n}
$$

is a linear function, whose characteristic is that every variable is raised to the first power only. A quadratic function, on the other hand, involves first and second powers of one or more independent variables, but the sum of
exponents of the variables appearing in any single term must not exceed two.

Example 2.6.1 $y=a x^{2}+b x y+c y^{2}+d x+e y+f$ is a quadratic function.

### 2.7 Levels of Generality

In discussing the various types of function, we have without explicit notice introducing examples of functions that pertain to varying levels of generality. In certain instances, we have written functions in the form

$$
y=7, y=6 x+4, y=x^{2}-3 x+1(\text { etc. })
$$

Not only are these expressed in terms of numerical coefficients, but they also indicate specifically whether each function is constant, linear, or quadratic. In terms of graphs, each such function will give rise to a well-defined unique curve. In view of the numerical nature of these functions, the solutions of the model based on them will emerge as numerical values also. The drawback is that, if we wish to know how our analytical conclusion will change when a different set of numerical coefficients comes into effect , we must go through the reasoning process afresh each time. Thus, the result obtained from specific functions have very little generality.

On a more general level of discussion and analysis, there are functions in the form

$$
y=a, y=b x+a, y=c x^{2}+b x+a(\text { etc. })
$$

Since parameters are used, each function represents not a single curve but a whole family of curves. With parametric functions, the outcome of mathematical operations will also be in terms of parameters. These results are more general.

In order to attain an even higher level of generality, we may resort to
the general function statement $y=f(x)$, or $z=g(x, y)$. When expressed in this form, the functions is not restricted to being either linear, quadratic, exponential, or trigonometric - all of which are subsumed under the notation. The analytical result based on such a general formulation will therefore have the most general applicability.

## Chapter 3

## Equilibrium Analysis in Economics

### 3.1 The Meaning of Equilibrium

Equilibrium in economics refers to a state where there is no tendency to change. It can be defined differently depending on the context. In general, it means selecting the best option from a set of available choices based on a certain criterion. This analysis of equilibrium is often referred to as statics. However, it is important to note that just because something is in equilibrium does not mean it is desirable or ideal.

This chapter explores two common examples of equilibrium analysis in economics. The first is a microeconomic example of a market equilibrium, where supply and demand conditions determine the price of a commodity. The second is a macroeconomic example of the equilibrium in a Keynesian national income model, where consumption and investment patterns determine the equilibrium level of income. These two examples will be used throughout the course.

### 3.2 Partial Market Equilibrium - A Linear Model

In a static-equilibrium model, the goal is to find the values of the endogenous variables that satisfy the equilibrium conditions of the model.

### 3.3 Partial Market Equilibrium - A Linear Model

In a static-equilibrium model, the standard problem is that of finding the set of values of the endogenous variables which will satisfy the equilibrium conditions of the model.

## Partial-Equilibrium Market Model

Partial-equilibrium market model is a model used to determine the price of a commodity in an isolated market. It considers three variables:
$Q_{d}=$ the quantity demanded of the commodity;
$Q_{s}=$ the quantity supplied of the commodity;
$P=$ the price of the commodity.

The Equilibrium Condition: $Q_{d}=Q_{s}$.
The model consists of equilibrium condition, demand function, and supply function:

$$
\begin{aligned}
Q_{d} & =Q_{s}, \\
Q_{d} & =a-b P \quad(a, b>0) \\
Q_{s} & =-c+d P \quad(c, d>0),
\end{aligned}
$$

$-b$ is the slope of $Q_{d}, a$ is the vertical intercept of $Q_{d}, d$ is the slope of $Q_{s}$, and $-c$ is the vertical intercept of $Q_{s}$.

Note that, contrary to the usual practice, quantity rather than price has been plotted vertically in the figure.


Figure 3.1: The linear model and its market equilibrium.

One way of finding the equilibrium is by successive elimination of variables and equations through substitution.

From $Q_{s}=Q_{d}$, we have

$$
a-b P=-c+d P
$$

and thus

$$
(b+d) P=a+c .
$$

Since $b+d \neq 0$, the equilibrium price is

$$
\bar{P}=\frac{a+c}{b+d} .
$$

The equilibrium quantity can be obtained by substituting $\bar{P}$ into either $Q_{s}$ or $Q_{d}$ :

$$
\bar{Q}=\frac{a d-b c}{b+d} .
$$

Since the denominator $(b+d)$ is positive, the positivity of $\bar{Q}$ requires that the numerator $(a d-b c)>0$. Thus, to be economically meaningful, the model should contain the additional restriction that $a d>b c$.

### 3.4 Partial Market Equilibrium - A Nonlinear Model

The partial market model can be nonlinear. For example, suppose the model is given by

$$
\begin{aligned}
Q_{d} & =Q_{s} \quad \text { (equilibrium condition); } \\
Q_{d} & =4-P^{2} \\
Q_{s} & =4 P-1
\end{aligned}
$$

As previously stated, this system of three equations can be reduced to a single equation by substitution.

$$
4-P^{2}=4 P-1
$$

or

$$
P^{2}+4 P-5=0,
$$

which is a quadratic equation. In general, given a quadratic equation in the form

$$
a x^{2}+b x+c=0 \quad(a \neq 0)
$$

its two roots can be obtained from the quadratic formula:

$$
\bar{x}_{1}, \bar{x}_{2}=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}
$$

where the " + " part of the " $\pm$ " sign yields $\bar{x}_{1}$ and " - " part yields $\bar{x}_{2}$. Thus, by applying the quadratic formulas to $P^{2}+4 P-5=0$, we have $\bar{P}_{1}=1$ and $\bar{P}_{2}=-5$, but only the first is economically admissible, as negative prices are ruled out.

## The Graphical Solution



Figure 3.2: The nonlinear model and its market equilibrium.

However, in general, the market model can be highly nonlinear, making it difficult or even impossible to find an explicit solution. In such cases, we may need to determine if there exists an implicit solution. This can be achieved using the Implicit-Function Theorem in Chapter 8.

### 3.5 General Market Equilibrium

So far, we have discussed the methods for analyzing an isolated market, where the demand and supply of a commodity are functions of the price
of that commodity alone. However, in reality, there are usually many substitutes and complementary goods, making it necessary to consider the effects not only of the price of the commodity itself but also of the prices of other related commodities.

A more realistic model for the demand and supply functions of a commodity should therefore take into account the prices of all related commodities, and the equilibrium condition for an $n$-commodity market model will involve $n$ equations, one for each commodity, in the form of:

$$
E_{i}=Q_{d i}-Q_{s i}=0 \quad \text { for } i=1,2, \ldots, n,
$$

where $Q_{d i}=Q_{d i}\left(P_{1}, P_{2}, \cdots, P_{n}\right)$ and $Q_{s i}=Q_{s i}\left(P_{1}, P_{2}, \cdots, P_{n}\right)$ are the demand and supply functions of commodity $i$, and $\left(P_{1}, P_{2}, \cdots, P_{n}\right)$ are prices of commodities.

Thus, solving $n$ equations for $\boldsymbol{P}=\left(P_{1}, P_{2}, \cdots, P_{n}\right)$ :

$$
E_{i}\left(P_{1}, P_{2}, \cdots, P_{n}\right)=0
$$

we obtain the $n$ equilibrium prices $\bar{P}_{i}-$ if a solution does indeed exist. And then the $\bar{Q}_{i}$ may be derived from the demand or supply functions.

## Two-Commodity Market Model

To illustrate the problem, let us consider a two-commodity market model with linear demand and supply functions. In parametric terms, such a model can be written as

$$
\begin{aligned}
& Q_{d 1}-Q_{s 1}=0 \quad \text { (equilibrium condition for commidity 1); } \\
& Q_{d 1}=a_{0}+a_{1} P_{1}+a_{2} P_{2} \quad \text { (consumer 1' demand function); } \\
& Q_{s 1}=b_{0}+b_{1} P_{1}+b_{2} P_{2} \quad \text { (producer 1' supply function); }
\end{aligned}
$$

$$
\begin{gathered}
Q_{d 2}-Q_{s 2}=0 \quad \text { (equilibrium condition for commdity 2); } \\
Q_{d 2}=\alpha_{0}+\alpha_{1} P_{1}+\alpha_{2} P_{2} \quad \text { (consumer 2' demand function); } \\
Q_{s 2}=\beta_{0}+\beta_{1} P_{1}+\beta_{2} P_{2} \quad \text { (producer 2' supply function). }
\end{gathered}
$$

By substituting the second and third equations into the first equation and the fifth and sixth equations into the fourth equation, the model can be reduced to two equations in two variables:

$$
\begin{aligned}
& \left(a_{0}-b_{0}\right)+\left(a_{1}-b_{1}\right) P_{1}+\left(a_{2}-b_{2}\right) P_{2}=0 \\
& \left(\alpha_{0}-\beta_{0}\right)+\left(\alpha_{1}-\beta_{1}\right) P_{1}+\left(\alpha_{2}-\beta_{2}\right) P_{2}=0
\end{aligned}
$$

If we let

$$
\begin{aligned}
c_{i} & =a_{i}-b_{i} \quad(i=0,1,2), \\
\gamma_{i} & =\alpha_{i}-\beta_{i}(i=0,1,2)
\end{aligned}
$$

the above two linear equations can be written as

$$
\begin{aligned}
& c_{1} P_{1}+c_{2} P_{2}=-c_{0} \\
& \gamma_{1} P_{1}+\gamma_{2} P_{2}=-\gamma_{0}
\end{aligned}
$$

which can be solved by further elimination of variables.
The solutions are

$$
\begin{aligned}
\bar{P}_{1} & =\frac{c_{2} \gamma_{0}-c_{0} \gamma_{2}}{c_{1} \gamma_{2}-c_{2} \gamma_{1}} ; \\
\bar{P}_{2} & =\frac{c_{0} \gamma_{1}-c_{1} \gamma_{0}}{c_{1} \gamma_{2}-c_{2} \gamma_{1}} .
\end{aligned}
$$

For these two values to make sense, certain restrictions should be imposed on the model. Firstly, we require the common denominator $c_{1} \gamma_{2}-$ $c_{2} \gamma_{1} \neq 0$. Secondly, to assure positivity, the numerator must have the same
sign as the denominator.

## Numerical Example

Suppose that the demand and supply functions are numerically as follows:

$$
\begin{gathered}
Q_{d 1}=10-2 P_{1}+P_{2} \\
Q_{s 1}=-2+3 P_{1} \\
Q_{d 2}=15+P_{1}-P_{2} \\
Q_{s 2}=-1+2 P_{2}
\end{gathered}
$$

By substitution, we have

$$
\begin{aligned}
& 5 P_{1}-P_{2}=12 \\
& -P_{1}+3 P_{2}=16
\end{aligned}
$$

which are two linear equations. The solutions for the equilibrium prices and quantities are $\bar{P}_{1}=52 / 14, \bar{P}_{2}=92 / 14, \bar{Q}_{1}=64 / 7, \bar{Q}_{2}=85 / 7$.

Similarly, for the $n$-commodities market model, when demand and supply functions are linear in prices, we can have $n$ linear equations. In the above, we assume that an equal number of equations and unknowns has a unique solution. However, some very simple examples should convince us that an equal number of equations and unknowns does not necessarily guarantee the existence of a unique solution.

For the two linear equations,

$$
\left\{\begin{array}{l}
x+y=8 \\
x+y=9
\end{array}\right.
$$

we can easily see that there is no solution.
The second example shows a system has an infinite number of solutions:

$$
\left\{\begin{array}{l}
2 x+y=12 \\
4 x+2 y=24
\end{array}\right.
$$

These two equations are functionally dependent, which means that one can be derived from the other. Consequently, one equation is redundant and may be dropped from the system. Any pair $(\bar{x}, \bar{y})$ is the solution as long as $(\bar{x}, \bar{y})$ satisfies $y=12-x$.

Now consider the case of more equations than unknowns. In general, there is no solution. But, when the number of unknowns equals the number of functionally independent equations, the solution exists and is unique. The following example shows this fact.

$$
\begin{gathered}
2 x+3 y=58 \\
y=18 \\
x+y=20
\end{gathered}
$$

Thus for simultaneous-equation model, we need systematic methods of testing the existence of a unique (or determinate) solution. There are our tasks in the following chapters.

### 3.6 Equilibrium in National-Income Analysis

The equilibrium analysis can be also applied to other areas of economics. As a simple example, we may cite the familiar Keynesian national-income
model,

$$
\begin{gathered}
Y=C+I_{0}+G_{0} \quad \text { (equilibrium condition); } \\
C=a+b Y \quad \text { (consumption function) }
\end{gathered}
$$

where $Y$ and $C$ stand for the endogenous variables national income and consumption expenditure, respectively, and $I_{0}$ and $G_{0}$ represent the exogenously determined investment and government expenditures, respectively.

Solving these two linear equations, we obtain the equilibrium national income and consumption expenditure:

$$
\begin{aligned}
\bar{Y} & =\frac{a+I_{0}+G_{0}}{1-b} \\
\bar{C} & =\frac{a+b\left(I_{0}+G_{0}\right)}{1-b}
\end{aligned}
$$

## Chapter 4

## Linear Models and Matrix

## Algebra

In the previous chapter, we saw that for the one-commodity partial market equilibrium model, the solutions for $\bar{P}$ and $\bar{Q}$ were relatively simple, despite the involvement of several parameters. However, as more commodities are incorporated into the model, these solution formulas quickly become cumbersome and unwieldy. Therefore, we need new methods that are suitable for handling a large system of simultaneous equations. Matrix algebra provides such a method.

Matrix algebra enables us to do many things, including:
(1) Providing a compact way of writing an equation system, even if it is extremely large.
(2) Leading to a way of testing the existence of a solution without actually solving it, by evaluating a determinant - a concept closely related to that of a matrix.
(3) Giving a method of finding that solution if it exists.

Throughout these lecture notes, we will use bold letters such as $\boldsymbol{a}$ to denote a vector and bold capital letters such as $\boldsymbol{A}$ to denote a matrix.

### 4.1 Matrix and Vectors

In general, a system of $m$ linear equations in $n$ variables $\left(x_{1}, x_{2}, \cdots, x_{n}\right)$ can be arranged into such formula

$$
\begin{align*}
& a_{11} x_{1}+a_{12} x_{2}+\cdots a_{1 n} x_{n}=d_{1}, \\
& a_{21} x_{1}+a_{22} x_{2}+\cdots a_{2 n} x_{n}=d_{2},  \tag{4.1.1}\\
& \cdots \\
& a_{m 1} x_{1}+a_{m 2} x_{2}+\cdots a_{m n} x_{n}=d_{m},
\end{align*}
$$

where the double-subscripted symbol $a_{i j}$ represents the coefficient appearing in the $i$ th equation and attached to the $j$ th variable $x_{j}$, and $d_{j}$ represents the constant term in the $j$ th equation.

Example 4.1.1 The two-commodity linear market model can be written after eliminating the quantity variables - as a system of two linear equations.

$$
\begin{aligned}
& c_{1} P_{1}+c_{2} P_{2}=-c_{0} \\
& \gamma_{1} P_{1}+\gamma_{2} P_{2}=-\gamma_{0}
\end{aligned}
$$

## Matrix as Arrays

There are essentially three types of ingredients in the equation system (3.1). The first is the set of coefficients $a_{i j}$; the second is the set of variables $x_{1}, x_{2}, \cdots, x_{n}$; and the last is the set of constant terms $d_{1}, d_{2}, \cdots, d_{m}$. If we arrange the three sets as three rectangular arrays and label them, respectively, by bold $\boldsymbol{A}, \boldsymbol{x}$, and $\boldsymbol{d}$, then we have

$$
\boldsymbol{A}=\left[\begin{array}{cccc}
a_{11} & a_{12} & \cdots & a_{1 n} \\
a_{21} & a_{22} & \cdots & a_{2 n} \\
\cdots & \cdots & \cdots & \cdots \\
a_{m 1} & a_{m 2} & \cdots & a_{m n}
\end{array}\right]
$$

$$
\begin{aligned}
& \boldsymbol{x}=\left(\begin{array}{c}
x_{1} \\
x_{2} \\
\ldots \\
x_{n}
\end{array}\right), \\
& \boldsymbol{d}=\left(\begin{array}{c}
d_{1} \\
d_{2} \\
\ldots \\
d_{m}
\end{array}\right) .
\end{aligned}
$$

Example 4.1.2 Given the linear-equation system:

$$
\begin{aligned}
& 6 x_{1}+3 x_{2}+x_{3}=22 \\
& x_{1}+4 x_{2}-2 x_{3}=12 \\
& 4 x_{1}-x_{2}+5 x_{3}=10
\end{aligned}
$$

we can write

$$
\begin{gathered}
\boldsymbol{A}=\left[\begin{array}{ccc}
6 & 3 & 1 \\
1 & 4 & -2 \\
4 & -1 & 5
\end{array}\right], \\
\boldsymbol{x}=\left(\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right), \\
\boldsymbol{d}=\left(\begin{array}{l}
22 \\
12 \\
10
\end{array}\right)
\end{gathered}
$$

Each of these three arrays given above constitutes a matrix.

A matrix is defined as a rectangular array of numbers, parameters, or variables. As a shorthand device, the array in matrix $\boldsymbol{A}$ can be written
more simple as

$$
\boldsymbol{A}=\left[a_{i j}\right]_{m \times n} \quad(i=1,2, \cdots, m ; j=1,2, \cdots, n) .
$$

## Vectors as Special Matrices

The number of rows and columns in a matrix together define the dimension of the matrix. For instance, $\boldsymbol{A}$ is said to be of dimension $m \times n$. In the special case where $m=n$, the matrix is called a square matrix.

If a matrix contains only one column (row), it is called a column (row) vector. For notation purposes, a row vector is often distinguished from a column vector by the use of a primed symbol:

$$
\boldsymbol{x}^{\prime}=\left[x_{1}, x_{2}, \cdots, x_{n}\right] .
$$

Remark 4.1.1 A vector is merely an ordered $n$-tuple and, as such, it may be interpreted as a point in an $n$-dimensional space.

Using the matrices defined in (3.2), we can express the equation system (3.1) simply as:

$$
\boldsymbol{A} \boldsymbol{x}=\boldsymbol{d}
$$

However, the equation $\boldsymbol{A} \boldsymbol{x}=\boldsymbol{d}$ prompts at least two questions. First, how do we multiply two matrices $\boldsymbol{A}$ and $\boldsymbol{x}$ ? Second, what is meant by the equality of $\boldsymbol{A} \boldsymbol{x}$ and $\boldsymbol{d}$ ? Since matrices involve whole blocks of numbers, the familiar algebraic operations defined for single numbers are not directly applicable, and there is a need for a new set of operation rules.

### 4.2 Matrix Operations

## The Equality of Two Matrices

$$
\boldsymbol{A}=\boldsymbol{B} \text { if and only if } a_{i j}=b_{i j} \text { for all } i=1,2, \cdots, m, j=1,2, \cdots, n .
$$

## Addition and Subtraction of Matrices

$$
\boldsymbol{A}+\boldsymbol{B}=\left[a_{i j}\right]+\left[b_{i j}\right]=\left[a_{i j}+b_{i j}\right],
$$

i.e., the addition of $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined as the addition of each pair of corresponding elements.

Remark 4.2.1 Two matrices can be added (equal) if and only if they have the same dimension.

## Example 4.2.1

$$
\left[\begin{array}{ll}
4 & 9 \\
2 & 1
\end{array}\right]+\left[\begin{array}{ll}
2 & 0 \\
0 & 7
\end{array}\right]=\left[\begin{array}{ll}
6 & 9 \\
2 & 8
\end{array}\right]
$$

Example 4.2.2

$$
\left[\begin{array}{lll}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23}
\end{array}\right]+\left[\begin{array}{lll}
b_{11} & b_{12} & b_{13} \\
b_{21} & b_{22} & b_{23}
\end{array}\right]=\left[\begin{array}{lll}
a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\
a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23}
\end{array}\right] .
$$

The Subtraction of Matrices:
$\boldsymbol{A}-\boldsymbol{B}$ is defined by

$$
\left[a_{i j}\right]-\left[b_{i j}\right]=\left[a_{i j}-b_{i j}\right] .
$$

## Example 4.2.3

$$
\left[\begin{array}{cc}
19 & 3 \\
2 & 0
\end{array}\right]-\left[\begin{array}{ll}
6 & 8 \\
1 & 3
\end{array}\right]=\left[\begin{array}{cc}
13 & -5 \\
1 & -3
\end{array}\right]
$$

## Scalar Multiplication:

$$
\lambda \boldsymbol{A}=\lambda\left[a_{i j}\right]=\left[\lambda a_{i j}\right],
$$

i.e., to multiply a matrix by a number is to multiply every element of that matrix by the given scalar.

## Example 4.2.4

$$
7\left[\begin{array}{cc}
3 & -1 \\
0 & 5
\end{array}\right]=\left[\begin{array}{cc}
21 & -7 \\
0 & 35
\end{array}\right]
$$

## Example 4.2.5

$$
-1\left[\begin{array}{lll}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23}
\end{array}\right]=\left[\begin{array}{lll}
-a_{11} & -a_{12} & -a_{13} \\
-a_{21} & -a_{22} & -a_{23}
\end{array}\right] .
$$

## Multiplication of Matrices:

Given two matrices $\boldsymbol{A}_{m \times n}$ and $\boldsymbol{B}_{p \times q}$, the conformability condition for multiplication $\boldsymbol{A B}$ is that the column dimension of $\boldsymbol{A}$ must be equal to the row dimension of $\boldsymbol{B}$, i.e., the matrix product $\boldsymbol{A} \boldsymbol{B}$ will be defined if and only if $n=p$. If defined, the product $\boldsymbol{A} \boldsymbol{B}$ will have the dimension $m \times q$.

The product $\boldsymbol{A} \boldsymbol{B}$ is defined by

$$
A B=C
$$

with $c_{i j}=a_{i 1} b_{1 j}+a_{i 2} b_{2 j}+\cdots+a_{i n} b_{n j}=\sum_{l=1}^{n} a_{i l} b_{l j}$.

## Example 4.2.6

$$
\left[\begin{array}{ll}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array}\right]\left[\begin{array}{ll}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{array}\right]=\left[\begin{array}{ll}
a_{11} b_{11}+a_{12} b_{21} & a_{11} b_{12}+a_{12} b_{22} \\
a_{21} b_{11}+a_{22} b_{21} & a_{21} b_{12}+a_{22} b_{22}
\end{array}\right] .
$$

## Example 4.2.7

$$
\left[\begin{array}{cc}
3 & 5 \\
4 & 6
\end{array}\right]\left[\begin{array}{cc}
-1 & 0 \\
4 & 7
\end{array}\right]=\left[\begin{array}{ll}
-3+20 & 35 \\
-4+24 & 42
\end{array}\right]=\left[\begin{array}{cc}
17 & 35 \\
20 & 42
\end{array}\right] .
$$

## Example 4.2.8

$$
\begin{gathered}
\boldsymbol{u}^{\prime}=\left[u_{1}, u_{2}, \cdots, u_{n}\right] \text { and } \boldsymbol{v}^{\prime}=\left[v_{1}, v_{2}, \cdots, v_{n}\right], \\
\boldsymbol{u}^{\prime} \boldsymbol{v}=u_{1} v_{1}+u_{2} v_{2}+\cdots+u_{n} v_{n}=\sum_{i=1}^{n} u_{i} v_{i} .
\end{gathered}
$$

This can be described by using the concept of the inner product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

$$
\boldsymbol{v} \cdot \boldsymbol{v}=u_{1} v_{1}+u_{2} v_{2}+\cdots+u_{n} v_{n}=\boldsymbol{u}^{\prime} \boldsymbol{v}
$$

Example 4.2.9 For the linear-equation system (4.1.1), the coefficient matrix and the variable vector are:

$$
\boldsymbol{A}=\left[\begin{array}{cccc}
a_{11} & a_{12} & \cdots & a_{1 n} \\
a_{21} & a_{22} & \cdots & a_{2 n} \\
\cdots & \cdots & \cdots & \cdots \\
a_{m 1} & a_{m 2} & \cdots & a_{m n}
\end{array}\right] \text { and } \boldsymbol{x}=\left[\begin{array}{c}
x_{1} \\
x_{2} \\
\cdots \\
x_{n}
\end{array}\right],
$$

and we then have

$$
\boldsymbol{A} \boldsymbol{x}=\left[\begin{array}{c}
a_{11} x_{1}+a_{12} x_{2}+\cdots+a_{1 n} x_{n} \\
a_{21} x_{1}+a_{22} x_{2}+\cdots+a_{2 n} x_{n} \\
\cdots \\
a_{m 1} x_{1}+a_{m 2} x_{2}+\cdots+a_{m n} x_{n}
\end{array}\right] .
$$

Thus, the linear-equation system (4.1.1) can indeed be simply written as

$$
\boldsymbol{A x}=\boldsymbol{d}
$$

Example 4.2.10 Given $\boldsymbol{u}=\left[\begin{array}{l}3 \\ 2\end{array}\right]$ and $\boldsymbol{v}^{\prime}=[1,4,5]$, we have

$$
\boldsymbol{u}^{\prime}=\left[\begin{array}{lll}
3 \times 1 & 3 \times 4 & 3 \times 5 \\
2 \times 1 & 2 \times 4 & 2 \times 5
\end{array}\right]=\left[\begin{array}{ccc}
3 & 12 & 15 \\
2 & 8 & 10
\end{array}\right]
$$

It is important to distinguish the meaning of $\boldsymbol{u} \boldsymbol{v}^{\prime}$ (a matrix with dimension $n \times n$ ) and $\boldsymbol{u}^{\prime} \boldsymbol{v}$ (a $1 \times 1$ matrix, or a scalar).

### 4.3 Linear (In)dependence of Vectors

Definition 4.3.1 A set of vectors $\boldsymbol{v}_{1}, \cdots, \boldsymbol{v}_{n}$ is said to be linearly dependent if one of them can be expressed as a linear combination of the remaining vectors, or equivalently, there exist scalars $k_{1}, \cdots, k_{n}$ not all zero such that

$$
\begin{equation*}
\sum_{i=1}^{n} k_{i} \boldsymbol{v}_{i}=\mathbf{0} \tag{4.3.2}
\end{equation*}
$$

otherwise, they are linearly independent (i.e., only when $k_{i}=0$ for all $i$ ).
Written in matrix form, we have

$$
\begin{equation*}
\boldsymbol{V} \boldsymbol{k}=\mathbf{0} \tag{4.3.3}
\end{equation*}
$$

where

$$
\boldsymbol{V}=\left[\begin{array}{c}
\boldsymbol{v}_{1}^{\prime} \\
\boldsymbol{v}_{2}^{\prime} \\
\ldots \\
\boldsymbol{v}_{m}^{\prime},
\end{array}\right]
$$

and $\mathbf{0}$ is the zero vector, i.e., all elements of $\mathbf{0}$ are zeros.
Note that if there are only two vectors, linear dependence means that one is a scalar multiple of the other.

Example 4.3.1 The three vectors

$$
\boldsymbol{v}_{1}=\left[\begin{array}{l}
2 \\
7
\end{array}\right], \boldsymbol{v}_{2}=\left[\begin{array}{l}
1 \\
8
\end{array}\right], \boldsymbol{v}_{3}=\left[\begin{array}{l}
4 \\
5
\end{array}\right]
$$

are linearly dependent because $\boldsymbol{v}_{3}$ is a linear combination of $\boldsymbol{v}_{1}$ and $\boldsymbol{v}_{2}$;

$$
3 \boldsymbol{v}_{1}-2 \boldsymbol{v}_{2}=\left[\begin{array}{c}
6 \\
21
\end{array}\right]-\left[\begin{array}{c}
2 \\
16
\end{array}\right]=\left[\begin{array}{l}
4 \\
5
\end{array}\right]=\boldsymbol{v}_{3}
$$

or

$$
3 \boldsymbol{v}_{1}-2 \boldsymbol{v}_{2}-\boldsymbol{v}_{3}=\mathbf{0}
$$

where $\mathbf{0}=\left[\begin{array}{l}0 \\ 0\end{array}\right]$ represents a zero vector.
Example 4.3.2 The three vectors

$$
\boldsymbol{v}_{1}=\left[\begin{array}{l}
2 \\
3
\end{array}\right], \boldsymbol{v}_{2}=\left[\begin{array}{l}
3 \\
1
\end{array}\right], \boldsymbol{v}_{3}=\left[\begin{array}{l}
1 \\
5
\end{array}\right]
$$

are linearly dependent since $\boldsymbol{v}_{1}$ is a linear combination of $\boldsymbol{v}_{2}$ and $\boldsymbol{v}_{3}$ :

$$
\boldsymbol{v}_{1}=\frac{1}{2} \boldsymbol{v}_{2}+\frac{1}{2} \boldsymbol{v}_{3} .
$$

In fact, as long as the number of vectors $n$ surpasses the dimension $m$ of the vector space they belong to, it can be shown that these vectors are linearly dependent.

Generally, verifying the linear (in)dependence of a set of $n$ vectors $\mathbf{v}_{1}, \mathbf{v}_{2}, \ldots, \mathbf{v}_{n}$ becomes challenging when $m$ exceeds 2 . Nevertheless, a way to simplify
this verification process involves checking if the system of homogeneous linear equations given by (4.3.3) admits a nonzero solution. We will provide a relatively simple way to check this.

### 4.4 Commutative, Associative, and Distributive Laws

The following basic laws on matrix operations sometimes can significantly simplify the computation of matrices.

The commutative and associative laws of matrices can be stated as follows:

## Commutative Law:

$$
\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}
$$

Proof: $\boldsymbol{A}+\boldsymbol{B}=\left[a_{i j}\right]+\left[b_{i j}\right]=\left[a_{i j}+b_{i j}\right]=\left[b_{i j}+a_{i j}\right]=\left[b_{i j}\right]+\left[a_{i j}\right]=\boldsymbol{B}+\boldsymbol{A}$.
Associative Law:

$$
(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C}) .
$$

Proof: $(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=\left(\left[a_{i j}\right]+\left[b_{i j}\right]\right)+\left[c_{i j}\right]=\left[a_{i j}+b_{i j}\right]+\left[c_{i j}\right]=\left[a_{i j}+b_{i j}+\right.$ $\left.c_{i j}\right]=\left[a_{i j}+\left(b_{i j}+c_{i j}\right)\right]=\left[a_{i j}\right]+\left(\left[b_{i j}+c_{i j}\right]\right)=\left[a_{i j}\right]+\left(\left[b_{i j}\right]+\left[c_{i j}\right]\right)=\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})$.

## Matrix Multiplication

Matrix multiplication is not commutative, that is,

$$
A B \neq B A
$$

Even when $\boldsymbol{A} \boldsymbol{B}$ is defined, $\boldsymbol{B} \boldsymbol{A}$ may not be; but even if both products are defined, $\boldsymbol{A} \boldsymbol{B}=\boldsymbol{B} \boldsymbol{A}$ may not hold.

Example 4.4.1 Let $\boldsymbol{A}=\left[\begin{array}{ll}1 & 2 \\ 3 & 4\end{array}\right], \boldsymbol{B}=\left[\begin{array}{cc}0 & -1 \\ 6 & 7\end{array}\right]$. Then $\boldsymbol{A} \boldsymbol{B}=\left[\begin{array}{ll}12 & 13 \\ 24 & 25\end{array}\right]$, but $\boldsymbol{B} \boldsymbol{A}=\left[\begin{array}{cc}-3 & -4 \\ 27 & 40\end{array}\right]$.

The scalar multiplication of a matrix does obey. The commutative law:

$$
k \boldsymbol{A}=\boldsymbol{A} k
$$

if $k$ is a scalar.
Associative Law:

$$
(A B) C=A(B C)
$$

provided $\boldsymbol{A}$ is $m \times n, \boldsymbol{B}$ is $n \times p$, and $\boldsymbol{C}$ is $p \times q$.

## Distributive Law

$$
\begin{gathered}
\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C})=\boldsymbol{A} \boldsymbol{B}+\boldsymbol{A} \boldsymbol{C} \quad[\text { premultiplication by } \boldsymbol{A}] ; \\
(\boldsymbol{B}+\boldsymbol{C}) \boldsymbol{A}=\boldsymbol{B} \boldsymbol{A}+\boldsymbol{C} \boldsymbol{A} \quad[\text { postmultiplication by } \boldsymbol{A}] .
\end{gathered}
$$

### 4.5 Identity Matrices and Null Matrices

Definition 4.5.1 Identity matrix, denoted by $\boldsymbol{I}$ or $\boldsymbol{I}_{n}$ in which $n$ indicates its dimension, is a square matrix with ones in its principal diagonal and zeros everywhere else. That is,

$$
\boldsymbol{I}_{n}=\left[\begin{array}{cccc}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 1
\end{array}\right]
$$

Fact 1: Given an $m \times n$ matrix $\boldsymbol{A}$, we have

$$
\boldsymbol{I}_{m} \boldsymbol{A}=\boldsymbol{A} \boldsymbol{I}_{n}=\boldsymbol{A}
$$

## Fact 2:

$$
\boldsymbol{A}_{m \times n} \boldsymbol{I}_{n} \boldsymbol{B}_{n \times p}=(\boldsymbol{A I}) \boldsymbol{B}=\boldsymbol{A} \boldsymbol{B} .
$$

## Fact 3:

$$
\left(\boldsymbol{I}_{n}\right)^{k}=\boldsymbol{I}_{n} .
$$

Idempotent Matrices: A matrix $\boldsymbol{A}$ is said to be idempotent if $\boldsymbol{A} \boldsymbol{A}=\boldsymbol{A}$.
Null Matrices: A null-or zero matrix-denoted by 0, plays the role of the number 0 . A null matrix is simply a matrix whose elements are all zero. Unlike $I$, the zero matrix is not restricted to being square. Null matrices obey the following rules of operation.

$$
\begin{gathered}
\boldsymbol{A}_{m \times n}+\mathbf{0}_{m \times n}=\boldsymbol{A}_{m \times n} ; \\
\boldsymbol{A}_{m \times n} \mathbf{0}_{n \times p}=\mathbf{0}_{m \times p} ; \\
\mathbf{0}_{q \times m} \boldsymbol{A}_{m \times n}=\mathbf{0}_{q \times n} .
\end{gathered}
$$

Remark 4.5.1 (a) $\boldsymbol{C D}=\boldsymbol{C E}$ does not imply $\boldsymbol{D}=\boldsymbol{E}$. For instance, for

$$
\boldsymbol{C}=\left[\begin{array}{ll}
2 & 3 \\
6 & 9
\end{array}\right], \boldsymbol{D}=\left[\begin{array}{ll}
1 & 1 \\
1 & 2
\end{array}\right], \boldsymbol{E}=\left[\begin{array}{cc}
-2 & 1 \\
3 & 2
\end{array}\right],
$$

we have

$$
\boldsymbol{C D}=\boldsymbol{C} \boldsymbol{E}=\left[\begin{array}{cc}
5 & 8 \\
15 & 24
\end{array}\right]
$$

even though $\boldsymbol{D} \neq \boldsymbol{E}$.
Then a question is: Under what condition, does $\boldsymbol{C D}=\boldsymbol{C} \boldsymbol{E}$ imply $\boldsymbol{D}=\boldsymbol{E}$ ? We will show that it is so if $\boldsymbol{C}$ has the inverse that we will discuss
shortly.
(b) Even if $\boldsymbol{A}$ and $\boldsymbol{B} \neq \mathbf{0}$, we can still have $\boldsymbol{A B}=\mathbf{0}$. Again, we will see this is not true if $\boldsymbol{A}$ or $\boldsymbol{B}$ has the inverse.

Example 4.5.1 $\boldsymbol{A}=\left[\begin{array}{ll}2 & 4 \\ 1 & 2\end{array}\right], \boldsymbol{B}=\left[\begin{array}{cc}-2 & 4 \\ 1 & -2\end{array}\right]$.
We have $\boldsymbol{A B}=\mathbf{0}$.

### 4.6 Transposes and Inverses

The transpose of a matrix $\boldsymbol{A}$ is obtained by interchanging the rows and columns of the matrix $\boldsymbol{A}$, resulting in a matrix of size $n \times m$. We define the transpose of $A$ as follows:

Formally, we have

Definition 4.6.1 A matrix $\boldsymbol{B}=\left[b_{i j}\right]_{n \times m}$ is said to be the transpose of $\boldsymbol{A}=$ $\left[a_{i j}\right]_{m \times n}$ if $a_{j i}=b_{i j}$ for all $i=1, \cdots, n$ and $j=1, \cdots, m$.

Usually transpose is denoted by $\boldsymbol{A}^{\prime}$ or $\boldsymbol{A}^{T}$.
Recipe - How to Find the Transpose of a Matrix:
The transpose $\boldsymbol{A}^{\prime}$ of $\boldsymbol{A}$ is obtained by making the columns of $\boldsymbol{A}$ into the rows of $\boldsymbol{A}^{\prime}$.

Example 4.6.1 For $\boldsymbol{A}=\left[\begin{array}{ccc}3 & 8 & -9 \\ 1 & 0 & 4\end{array}\right]$, its transpose is

$$
\boldsymbol{A}^{\prime}=\left[\begin{array}{cc}
3 & 1 \\
8 & 0 \\
-9 & 4
\end{array}\right]
$$

Thus, by definition, if the dimension of a matrix $\boldsymbol{A}$ is $m \times n$, then the dimension of its transpose $\boldsymbol{A}^{\prime}$ must be $n \times m$.

Example 4.6.2 For

$$
\boldsymbol{D}=\left[\begin{array}{lll}
1 & 0 & 4 \\
0 & 3 & 7 \\
4 & 7 & 2
\end{array}\right]
$$

its transpose is:

$$
\boldsymbol{D}^{\prime}=\left[\begin{array}{lll}
1 & 0 & 4 \\
0 & 3 & 7 \\
4 & 7 & 2
\end{array}\right]=\boldsymbol{D}
$$

Definition 4.6.2 A matrix $\boldsymbol{A}$ is said to be symmetric if $\boldsymbol{A}^{\prime}=\boldsymbol{A}$.
A matrix $\boldsymbol{A}$ is called anti-symmetric (or skew-symmetric) if $\boldsymbol{A}^{\prime}=-\boldsymbol{A}$. A matrix $\boldsymbol{A}$ is called orthogonal if $\boldsymbol{A}^{\prime} \boldsymbol{A}=\boldsymbol{I}$.

## Properties of Transposes:

a) $\left(\boldsymbol{A}^{\prime}\right)^{\prime}=\boldsymbol{A}$;
b) $(\boldsymbol{A}+\boldsymbol{B})^{\prime}=\boldsymbol{A}^{\prime}+\boldsymbol{B}^{\prime}$;
c) $(\alpha \boldsymbol{A})^{\prime}=\alpha \boldsymbol{A}^{\prime}$ where $\alpha$ is a real number;
d) $(\boldsymbol{A B})^{\prime}=\boldsymbol{B}^{\prime} \boldsymbol{A}^{\prime}$.

The property d) states that the transpose of a product is the product of the transposes in reverse order.

## Inverses and Their Properties

For a given square matrix $\boldsymbol{A}$, while its transpose $\boldsymbol{A}^{\prime}$ is always derivable, its inverse matrix may or may not exist.

Definition 4.6.3 A matrix, denoted by $\boldsymbol{A}^{-1}$, is the inverse of $\boldsymbol{A}$ if the following conditions are satisfied:
(1) $\boldsymbol{A}$ is a square matrix;
(2) $\boldsymbol{A} \boldsymbol{A}^{-1}=\boldsymbol{A}^{-1} \boldsymbol{A}=\boldsymbol{I}$.

If a square matrix $\boldsymbol{A}$ has an inverse, $\boldsymbol{A}$ is said to be nonsingular, whereas if $\boldsymbol{A}$ has no inverse, it is said to be a singular matrix.

Here are some important properties of inverse matrices:

1. Not every square matrix has an inverse, and therefore, squareness is a necessary, but not sufficient, condition for the existence of an inverse.
2. If $\boldsymbol{A}$ is nonsingular, then $\boldsymbol{A}$ and $\boldsymbol{A}^{-1}$ are inverse of each other, i.e., $\left(\boldsymbol{A}^{-1}\right)^{-1}=\boldsymbol{A}$.
3. If $\boldsymbol{A}$ is $n \times n$, then $\boldsymbol{A}^{-1}$ is also $n \times n$.
4. The inverse of $\boldsymbol{A}$ is unique.

Proof. Let $\boldsymbol{B}$ and $\boldsymbol{C}$ both be inverses of $\boldsymbol{A}$. Then

$$
B=B I=B A C=I C=C
$$

5. $\boldsymbol{A} \boldsymbol{A}^{-1}=I$ implies that $\boldsymbol{A}^{-1} \boldsymbol{A}=I$.

Proof. We need to show that if $\boldsymbol{A} \boldsymbol{A}^{-1}=\boldsymbol{I}$, and if there is a matrix $\boldsymbol{B}$ such that $\boldsymbol{B} \boldsymbol{A}=\boldsymbol{I}$, then $\boldsymbol{B}=\boldsymbol{A}^{-1}$. To see this, postmultiplying both sides of $\boldsymbol{B} \boldsymbol{A}=\boldsymbol{I}$ by $\boldsymbol{A}^{-1}$, we have $\boldsymbol{B} \boldsymbol{A} \boldsymbol{A}^{-1}=\boldsymbol{A}^{-1}$ and thus $\boldsymbol{B}=\boldsymbol{A}^{-1}$.
6. Suppose that $\boldsymbol{A}$ and $\boldsymbol{B}$ are nonsingular matrices with dimension $n \times$ $n$.
(a) $(\boldsymbol{A B})^{-1}=\boldsymbol{B}^{-1} \boldsymbol{A}^{-1}$
(b) $\left(\boldsymbol{A}^{\prime}\right)^{-1}=\left(\boldsymbol{A}^{-1}\right)^{\prime}$

It's worth noting that while the transpose of a matrix always exists, the inverse does not necessarily exist.

Example 4.6.3 Let $\boldsymbol{A}=\left[\begin{array}{ll}3 & 1 \\ 0 & 2\end{array}\right]$ and $\boldsymbol{B}=\frac{1}{6}\left[\begin{array}{cc}2 & -1 \\ 0 & 3\end{array}\right]$. Then

$$
\boldsymbol{A} \boldsymbol{B}=\left[\begin{array}{ll}
3 & 1 \\
0 & 2
\end{array}\right]\left[\begin{array}{cc}
2 & -1 \\
0 & 3
\end{array}\right] \frac{1}{6}=\left[\begin{array}{ll}
6 & \\
& 6
\end{array}\right] \frac{1}{6}=\left[\begin{array}{ll}
1 & \\
& 1
\end{array}\right] .
$$

So $\boldsymbol{B}$ is the inverse of $\boldsymbol{A}$.

## Inverse Matrix and Solution of Linear-Equation System

The application of the concept of inverse matrix to the solution of a simultaneous linear-equation system is immediate and direct. Consider

$$
A x=d
$$

If $\boldsymbol{A}$ is a nonsingular matrix, then premultiplying both sides of $\boldsymbol{A x}=\boldsymbol{d}$ by $\boldsymbol{A}^{-1}$, we have

$$
\boldsymbol{A}^{-1} \boldsymbol{A} \boldsymbol{x}=\boldsymbol{A}^{-1} \boldsymbol{d}
$$

So, $\boldsymbol{x}=\boldsymbol{A}^{-1} \boldsymbol{d}$ is the solution of $\boldsymbol{A} \boldsymbol{x}=\boldsymbol{d}$, and furthermore, the solution is unique since $\boldsymbol{A}^{-1}$ is unique. Methods of testing the existence of the inverse and its calculation will be discussed in the next chapter.

## Chapter 5

## Linear Models and Matrix Algebra (Continued)

In chapter 4, it was shown that a linear-equation system can be written in a compact notation. Moreover, such an equation system can be solved by finding the inverse of the coefficient matrix, provided the inverse exists. This chapter studies how to test for the existence of the inverse, how to find that inverse, and consequently gives ways of solving linear equation systems.

### 5.1 Conditions for Nonsingularity of a Matrix

As pointed out earlier, the squareness condition is necessary but not sufficient for the existence of the inverse $\boldsymbol{A}^{-1}$ of a matrix $\boldsymbol{A}$. What are the conditions for the existence of the inverse $\boldsymbol{A}^{-1}$ of a matrix $\boldsymbol{A}$ ?

## Conditions for Nonsingularity

When the squareness condition is already met, a sufficient condition for the nonsingularity of a matrix is that its rows (or equivalently, its columns)
are linearly independent. In fact, the necessary and sufficient conditions for nonsingularity are that the matrix satisfies the squareness and linear independence conditions.

To see this, write an $n \times n$ coefficient matrix $\boldsymbol{A}$ as an ordered set of row vectors:

$$
\boldsymbol{A}=\left[\begin{array}{ccc}
a_{11} & a_{12} & \cdots a_{1 n} \\
a_{21} & a_{22} & \cdots a_{2 n} \\
\cdots & \cdots & \cdots \\
a_{n 1} & a_{n 2} & \cdots a_{n n}
\end{array}\right]=\left[\begin{array}{c}
\boldsymbol{v}_{1}^{\prime} \\
\boldsymbol{v}_{2}^{\prime} \\
\cdots \\
\boldsymbol{v}_{n}^{\prime}
\end{array}\right]
$$

where $\boldsymbol{v}_{i}^{\prime}=\left[a_{i 1}, a_{i 2}, \cdots, a_{i n}\right], i=1,2, \cdots, n$. As we discussed in the previous chapter, for the rows of a matrix $\boldsymbol{A}$ to be linearly independent, for any set of scalars $k_{i}, \sum_{i=1}^{n} k_{i} \boldsymbol{v}_{i}=0$ if and only if $k_{i}=0$ for all $i$, which is equivalent to the homogeneous linear-equation system $\boldsymbol{A} \boldsymbol{k}=\mathbf{0}$ having the unique solution $\boldsymbol{k}=\mathbf{0}$, where its transpose $\boldsymbol{k}^{\prime}=\left(k_{1}, k_{2}, \ldots, k_{n}\right)$.

This is true when $\boldsymbol{A}$ has the inverse, which is true if and only if the matrix satisfies the squareness and linear independence conditions.

Example 5.1.1 For a given matrix,

$$
\left[\begin{array}{ccc}
3 & 4 & 5 \\
0 & 1 & 2 \\
6 & 8 & 10
\end{array}\right],
$$

since $\boldsymbol{v}_{3}^{\prime}=2 v_{1}^{\prime}+0 \boldsymbol{v}_{2}^{\prime}$, so the matrix is singular.
Example 5.1.2 $\boldsymbol{B}=\left[\begin{array}{ll}1 & 2 \\ 3 & 4\end{array}\right]$ is nonsingular since their two rows are not proportional.

Example 5.1.3 $\boldsymbol{C}=\left[\begin{array}{cc}-2 & 1 \\ 6 & -3\end{array}\right]$ is singular their two rows are proportional.

## Rank of a Matrix

The above discussion on row linear independence are regard to square matrices, it is equally applicable to any $m \times n$ rectangular matrix.

Definition 5.1.1 A matrix $\boldsymbol{A}_{m \times n}$ is said to be of rank $\gamma$ if the maximum number of linearly independent rows that can be found in such a matrix is $\gamma$.

By definition, an $n \times n$ nonsingular matrix $\boldsymbol{A}$ has $n$ linearly independent rows (or columns); consequently it must be of rank $n$. Conversely, an $n \times n$ matrix having rank $n$ must be nonsingular.

### 5.2 Test of Nonsingularity by Use of Determinant

To determine whether a square matrix is nonsingular by finding the inverse of the matrix is not an easy job. However, we can use the determinant of the matrix to easily determine if a square matrix is nonsingular.

## Determinant and Nonsingularity

The determinant of a square matrix $\boldsymbol{A}$, denoted by $|\boldsymbol{A}|$, is a uniquely defined scalar associated with that matrix. Determinants are defined only for square matrices. For a $2 \times 2$ matrix:

$$
\boldsymbol{A}=\left[\begin{array}{ll}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array}\right],
$$ its determinant is defined as follows:

$$
|\boldsymbol{A}|=\left|\begin{array}{ll}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array}\right|=a_{11} a_{22}-a_{12} a_{21}
$$

In view of the dimension of matrix $\boldsymbol{A},|\boldsymbol{A}|$ as defined in the above is known as a second-order determinant.

Example 5.2.1 Given $\boldsymbol{A}=\left[\begin{array}{cc}10 & 4 \\ 8 & 5\end{array}\right]$ and $\boldsymbol{B}=\left[\begin{array}{cc}3 & 5 \\ 0 & -1\end{array}\right]$, then

$$
\begin{gathered}
|\boldsymbol{A}|=\left|\begin{array}{cc}
10 & 4 \\
8 & 5
\end{array}\right|=50-32=18 \\
|\boldsymbol{B}|=\left|\begin{array}{cc}
3 & 5 \\
0 & -1
\end{array}\right|=-3-5 \times 0=-3 .
\end{gathered}
$$

Example 5.2.2 $\boldsymbol{A}=\left[\begin{array}{cc}2 & 6 \\ 8 & 24\end{array}\right]$. Then its determinant is

$$
|\boldsymbol{A}|=\left|\begin{array}{cc}
2 & 6 \\
8 & 24
\end{array}\right|=2 \times 24-6 \times 8=48-48=0
$$

This example illustrates the fact that the determinant of a matrix is equal to zero if and only if its rows are linearly dependent. As we will see, the value of a determinant $|\boldsymbol{A}|$ can serve as a criterion for testing the linear independence of the rows (hence nonsingularity) of matrix $\boldsymbol{A}$, but it can also be used as an input in the calculation of the inverse $\boldsymbol{A}^{-1}$, if it exists.

## Evaluating a Third-Order Determinant

For a $3 \times 3$ matrix $\boldsymbol{A}$, its third-order determinants have the value

$$
\begin{aligned}
|\boldsymbol{A}| & =\left|\begin{array}{ccc}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{array}\right|=a_{11}\left|\begin{array}{cc}
a_{22} & a_{23} \\
a_{32} & a_{33}
\end{array}\right|-a_{12}\left|\begin{array}{cc}
a_{21} & a_{23} \\
a_{31} & a_{33}
\end{array}\right|+a_{13}\left|\begin{array}{cc}
a_{21} & a_{22} \\
a_{31} & a_{32}
\end{array}\right| \\
& =a_{11} a_{22} a_{33}-a_{11} a_{23} a_{32}-a_{12} a_{21} a_{33}+a_{12} a_{23} a_{31}+a_{13} a_{21} a_{32}-a_{13} a_{22} a_{31} .
\end{aligned}
$$

We can use the following diagram to calculate the third-order determinant.


Figure 5.1: The graphic illustration for calculating the third-order determinant.

## Example 5.2.3

$$
\begin{aligned}
\left|\begin{array}{lll}
2 & 1 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{array}\right| & =2 \times 5 \times 9+1 \times 6 \times 7+4 \times 8 \times 3-3 \times 5 \times 7-1 \times 4 \times 9-6 \times 8 \times 2 \\
& =90+42+96-105-36-96=-9
\end{aligned}
$$

## Example 5.2.4

$$
\begin{aligned}
\left|\begin{array}{lll}
0 & 1 & 2 \\
3 & 4 & 5 \\
6 & 7 & 8
\end{array}\right| & =0 \times 4 \times 8+1 \times 5 \times 6+3 \times 7 \times 2-2 \times 4 \times 6-1 \times 3 \times 8-5 \times 7 \times 0 \\
& =0+30+42-48-24-0=0
\end{aligned}
$$

## Example 5.2.5

$$
\begin{aligned}
\left|\begin{array}{ccc}
-1 & 2 & 1 \\
0 & 3 & 2 \\
1 & 0 & 2
\end{array}\right| & =-1 \times 3 \times 2+2 \times 2 \times 1+0 \times 0 \times 1-1 \times 3 \times 1-2 \times 0 \times 2-2 \times 0 \times(-1) \\
& =-6+4+0-3-0-0=-5 .
\end{aligned}
$$

The method of cross-diagonal multiplication provides a handy way of evaluating a third-order determinant, but unfortunately it is not applicable to determinants of orders higher than 3. For the latter, we must resort to the so-called "Laplace expansion" of the determinant.

## Evaluating an $n$ th-Order Determinant by Laplace Expansion

The minor of the element $a_{i j}$ of a determinant $|\boldsymbol{A}|$, denoted by $\left|\boldsymbol{M}_{i j}\right|$, can be obtained by deleting the $i$ th row and $j$ th column of the determinant $|\boldsymbol{A}|$.

For instance, for a third determinant, the minors of $a_{11}, a_{12}$ and $a_{13}$ are

$$
\left|\boldsymbol{M}_{11}\right|=\left|\begin{array}{ll}
a_{22} & a_{23} \\
a_{32} & a_{33}
\end{array}\right|,\left|\boldsymbol{M}_{12}\right|=\left|\begin{array}{cc}
a_{21} & a_{23} \\
a_{31} & a_{33}
\end{array}\right|,\left|\boldsymbol{M}_{13}\right|=\left|\begin{array}{cc}
a_{21} & a_{21} \\
a_{31} & a_{32}
\end{array}\right| .
$$

A concept closely related to the minor is that of the cofactor. A cofactor, denoted by $\boldsymbol{C}_{i j}$, is a minor with a prescribed algebraic sign attached to it. Formally, it is defined by

$$
\left|\boldsymbol{C}_{i j}\right|=(-1)^{i+j}\left|\boldsymbol{M}_{i j}\right|=\left\{\begin{array}{cl}
-\left|\boldsymbol{M}_{i j}\right| & \text { if } i+j \text { is odd } \\
\left|\boldsymbol{M}_{i j}\right| & \text { if } i+j \text { is even. }
\end{array}\right.
$$

Thus, if the sum of the two subscripts $i$ and $j$ in $\boldsymbol{M}_{i j}$ is even, then $\left|\boldsymbol{C}_{i j}\right|=\left|\boldsymbol{M}_{i j}\right|$. If it is odd, then $\left|\boldsymbol{C}_{i j}\right|=-\left|\boldsymbol{M}_{i j}\right|$.

Using these new concepts, we can express a third-order determinant as

$$
\begin{aligned}
|\boldsymbol{A}| & =a_{11}\left|\boldsymbol{M}_{11}\right|-a_{12}\left|\boldsymbol{M}_{12}\right|+a_{13}\left|\boldsymbol{M}_{13}\right| \\
& =a_{11}\left|\boldsymbol{C}_{11}\right|+a_{12}\left|\boldsymbol{C}_{12}\right|+a_{13}\left|\boldsymbol{C}_{13}\right| .
\end{aligned}
$$

The Laplace expansion of a third-order determinant simplifies the evaluation problem by reducing it to the calculation of specific second-order determinants. In a broader context, the Laplace expansion of an nth-order determinant simplifies the task of evaluating $n$ cofactors, each of which is of order $(n-1)$. Repeatedly applying this process systematically decreases the order of determinants until eventually reaching the fundamental second-order determinants. At this point, the original determinant's value becomes straightforward to calculate.

Formally, the value of a determinant $|\boldsymbol{A}|$ of order $n$ can be found by the

Laplace expansion of any row or any column as follows:

$$
\begin{aligned}
|\boldsymbol{A}| & =\sum_{j=1}^{n} a_{i j}\left|\boldsymbol{C}_{i j}\right| \quad[\text { expansion by the } i \text { th row }] \\
& =\sum_{i=1}^{n} a_{i j}\left|\boldsymbol{C}_{i j}\right| \quad[\text { expansion by the } j \text { th column }] .
\end{aligned}
$$

Even though one can expand $|\boldsymbol{A}|$ by any row or any column, as the numerical calculation is concerned, a row or column with largest number of 0's or 1's is always preferable for this purpose, because a 0 times its cofactor is simply 0 .

Example 5.2.6 For the $|\boldsymbol{A}|=\left|\begin{array}{ccc}5 & 6 & 1 \\ 2 & 3 & 0 \\ 7 & -3 & 0\end{array}\right|$, the easiest way to expand the determinant is by the third column, which consists of the elements 1,0 , and 0 . Thus,

$$
|\boldsymbol{A}|=1 \times(-1)^{1+3}\left|\begin{array}{cc}
2 & 3 \\
7 & -3
\end{array}\right|=-6-21=-27
$$

## Example 5.2.7

$$
\left|\begin{array}{llll}
0 & 0 & 0 & 1 \\
0 & 0 & 2 & 0 \\
0 & 3 & 0 & 0 \\
4 & 0 & 0 & 0
\end{array}\right|=1 \times(-1)^{1+4}\left|\begin{array}{lll}
0 & 0 & 2 \\
0 & 3 & 0 \\
4 & 0 & 0
\end{array}\right|=-1 \times(-24)=24
$$

A triangular matrix is a special type of square matrix. A square matrix is called the lower triangular if all the entries above the main diagonal are zero. Similarly, a square matrix is called the upper triangular if all the entries below the main diagonal are zero.

Example 5.2.8 (Upper Triangular Determinant) This example shows that the value of an upper triangular determinant is the product of all elements on the main diagonal.

$$
\begin{aligned}
\left|\begin{array}{cccc}
a_{11} & a_{12} & \cdots & a_{1 n} \\
0 & a_{22} & \cdots & a_{2 n} \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & a_{n n}
\end{array}\right| & =a_{11} \times(-1)^{1+1}\left|\begin{array}{cccc}
a_{22} & a_{23} & \cdots & a_{2 n} \\
0 & a_{33} & \cdots & a_{3 n} \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & a_{n n}
\end{array}\right| \\
& =a_{11} \times a_{22} \times(-1)^{1+1}\left|\begin{array}{cccc}
a_{33} & a_{34} & \cdots & a_{3 n} \\
0 & a_{44} & \cdots & a_{4 n} \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & a_{n n}
\end{array}\right| \\
& =\cdots=a_{11} \times a_{22} \times a_{n n} .
\end{aligned}
$$

### 5.3 Basic Properties of Determinants

Property I. The determinant of a matrix $A$ has the same value as that of its transpose $A^{\prime}$, i.e.,

$$
|\boldsymbol{A}|=\left|\boldsymbol{A}^{\prime}\right| .
$$

Example 5.3.1 For

$$
|\boldsymbol{A}|=\left|\begin{array}{ll}
a & b \\
c & d
\end{array}\right|=a d-b c,
$$

we have

$$
\left|\boldsymbol{A}^{\prime}\right|=\left|\begin{array}{ll}
a & c \\
b & d
\end{array}\right|=a d-b c=|\boldsymbol{A}|
$$

Property II. The interchange of any two rows (or any two columns) will alter the sign, but not the numerical value of the determinant.

Example 5.3.2 $\left|\begin{array}{ll}a & b \\ c & d\end{array}\right|=a d-b c$, but the interchange of the two rows yields

$$
\left|\begin{array}{cc}
c & d \\
a & b
\end{array}\right|=b c-a d=-(a d-b c)
$$

Property III. The multiplication of any one row (or one column) by a scalar $k$ will change the value of the determinant $k$-fold, i.e., for $|\boldsymbol{A}|$,

$$
\left|\begin{array}{cccc}
a_{11} & a_{12} & \cdots & a_{1 n} \\
\cdots & \cdots & \cdots & \cdots \\
k a_{i 1} & k a_{i 2} & \cdots & k a_{i n} \\
\cdots & \cdots & \cdots & \cdots \\
a_{n 1} & a_{n 2} & \cdots & a_{n n}
\end{array}\right|=k\left|\begin{array}{cccc}
a_{11} & a_{12} & \cdots & a_{1 n} \\
\cdots & \cdots & \cdots & \cdots \\
a_{i 1} & a_{i 2} & \cdots & a_{i n} \\
\cdots & \cdots & \cdots & \cdots \\
a_{n 1} & a_{n 2} & \cdots & a_{n n}
\end{array}\right|=k|\boldsymbol{A}| .
$$

In contrast, the factoring of a matrix requires the presence of a common divisor for all its elements, as in

$$
k\left[\begin{array}{ccc}
a_{11} & a_{12} & \cdots \\
a_{21} & a_{22} & \cdots \\
a_{2 n} \\
\cdots & \cdots & \cdots \\
a_{n 1} & a_{n 2} & \cdots
\end{array}\right]=\left[\begin{array}{ccc}
k a_{11} & k a_{12} & \cdots k a_{1 n} \\
k a_{21} & k a_{22} & \cdots k a_{2 n} \\
\cdots & \cdots & \cdots \\
k a_{n 1} & k a_{n 2} & \cdots k a_{n n}
\end{array}\right] .
$$

Property IV. The addition (subtraction) of a multiple of any row (or column) to (from) another row (or column) will leave the value of the determinant unaltered.

This is an extremely useful property, which can be used to greatly simplify the computation of a determinant.

## Example 5.3.3

$$
\left|\begin{array}{cc}
a & b \\
c+k a & d+k b
\end{array}\right|=a(d+k b)-b(c+k a)=a d-b c=\left|\begin{array}{cc}
a & b \\
c & d
\end{array}\right| .
$$

## Example 5.3.4

$$
\begin{aligned}
\left|\begin{array}{llll}
a & b & b & b \\
b & a & b & b \\
b & b & a & b \\
b & b & b & a
\end{array}\right| & =\left|\begin{array}{llll}
a+3 b & b & b & b \\
a+3 b & a & b & b \\
a+3 b & b & a & b \\
a+3 b & b & b & a
\end{array}\right|=\left|\begin{array}{cccc}
1 & b & b & b \\
1 & a & b & b \\
1 & b & a & b \\
1 & b & b & a
\end{array}\right| \\
& =(a+3 b)\left|\begin{array}{cccc}
1 & b & b & b \\
0 & a-b & 0 & 0 \\
0 & 0 & a-b & 0 \\
0 & 0 & 0 & a-b
\end{array}\right|=(a+3 b)(a-b)^{3} .
\end{aligned}
$$

The second determinant in the above equation is obtained by adding the second column, the third column, and the fourth column to the first column, respectively. The third determinant is obtained by taking out the common factor $(a+3 b)$ from the first column. The fourth determinant is obtained by adding the negative of the first row to the second row, the third row, and the fourth row in the second determinant, respectively. Since the fourth determinant is upper triangular, its value is the product of all elements on the main diagonal.

Example 5.3.5 Similarly, we can compute the following example:

$$
\begin{aligned}
\left|\begin{array}{llll}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 1 \\
3 & 4 & 1 & 2 \\
4 & 1 & 2 & 3
\end{array}\right| & =\left|\begin{array}{llll}
10 & 2 & 3 & 4 \\
10 & 3 & 4 & 1 \\
10 & 4 & 1 & 2 \\
10 & 1 & 2 & 3
\end{array}\right|=\left|\begin{array}{cccc}
10 & 2 & 3 & 4 \\
0 & 1 & 1 & -3 \\
0 & 2 & -2 & -2 \\
0 & -1 & -1 & -1
\end{array}\right| \\
& =(-1)^{1+1} 10\left|\begin{array}{ccc}
1 & 1 & -3 \\
2 & -2 & -2 \\
-1 & -1 & -1
\end{array}\right|=10\left|\begin{array}{ccc}
1 & 1 & -3 \\
0 & -4 & 4 \\
0 & 0 & -4
\end{array}\right|=160
\end{aligned}
$$

Example 5.3.6 Conpute the following determinant:

$$
\begin{aligned}
\left|\begin{array}{cccc}
-2 & 5 & -1 & 3 \\
1 & -9 & 13 & 7 \\
3 & -1 & 5 & -5 \\
2 & 8 & -7 & -10
\end{array}\right| & =\left|\begin{array}{cccc}
0 & -13 & 25 & 17 \\
1 & -9 & 13 & 7 \\
0 & 26 & -34 & -26 \\
0 & 26 & -33 & -24
\end{array}\right| \\
& =(-1)^{1+2}\left|\begin{array}{ccc}
-13 & 25 & 17 \\
26 & -34 & -26 \\
26 & -33 & -24
\end{array}\right|=13\left|\begin{array}{ccc}
1 & 25 & 17 \\
-2 & -34 & -26 \\
-2 & -33 & -24
\end{array}\right| \\
& =13\left|\begin{array}{ccc}
1 & 25 & 17 \\
0 & 16 & 8 \\
0 & 17 & 10
\end{array}\right|=(-1)^{1+1} 13\left|\begin{array}{cc}
16 & 8 \\
17 & 10
\end{array}\right|=312 .
\end{aligned}
$$

In this calculation, the second determinant is obtained by adding 2 times the second row to the first row, -3 times the second row to the third row, and -2 times the second row to the fourth row. The third determinant is obtained by expanding the first column. The fourth determinant is obtained by factoring out -13 from the first column. The fifth determinant is obtained by adding 2 times the first row to both the second and third rows. The sixth determinant is obtained by expanding the first column.

Property V. If one row (or column) is a multiple of another row (or column), the value of the determinant will be zero.

## Example 5.3.7

$$
\left|\begin{array}{cc}
k a & k b \\
a & b
\end{array}\right|=k a b-k a b=0 .
$$

Remark 5.3.1 Property V is a logic consequence of Property IV.

Property VI. If $A$ and $B$ are both square matrices, then $|A B|=|A||B|$.
The aforementioned basic properties of determinants are useful in various ways. They can simplify the task of evaluating determinants. By adding or subtracting multipliers of one row (or column) from another, the elements of the determinant can be reduced to much simpler numbers. If we apply these properties to transform some row or column into a form containing mostly zeros or ones, Laplace expansion of the determinant will become a much more manageable task.

Property VII. $\left|\boldsymbol{A}^{-1}\right|=\frac{1}{|\boldsymbol{A}|}$. As a consequence, if $A^{-1}$ exists, we must have $|\boldsymbol{A}| \neq 0$. The converse is also true.

Recipe - How to Calculate the Determinant:

1. The multiplication of any one row (or column) by a scalar $k$ will change the value of the determinant $k$-fold.
2. The interchange of any two rows (columns) will change the sign but not the numerical value of the determinant.
3. If a multiple of any row is added to (or subtracted from) any other row it will not change the value or the sign of the determinant. The same holds true for columns (i.e. the determinant is not affected by linear operations with rows (or columns)).

## 56CHAPTER 5. LINEAR MODELS AND MATRIX ALGEBRA (CONTINUED)

4. If two rows (or columns) are proportional, i.e., they are linearly dependent, then the determinant will vanish.
5. The determinant of a triangular matrix is a product of its principal diagonal elements.

Using these rules, we can simplify the matrix (e.g., obtain as many zero elements as possible) and then apply Laplace expansion.

## Determinantal Criterion for Nonsingularity

Our primary concern here is to link the linear dependence of rows with the vanishing of a determinant. By Property I, we can easily see that row independence is equivalent to column independence.

Given a linear-equation system $\boldsymbol{A} \boldsymbol{x}=\boldsymbol{d}$, where $\boldsymbol{A}$ is an $n \times n$ coefficient matrix, we have

$$
\begin{aligned}
|\boldsymbol{A}| \neq 0 & \Leftrightarrow \boldsymbol{A} \text { is row (or column) independent } \\
& \Leftrightarrow \operatorname{rank}(\boldsymbol{A})=n \\
& \Leftrightarrow \boldsymbol{A} \text { is nonsingular } \\
& \Leftrightarrow \boldsymbol{A}^{-1} \text { exists } \\
& \Leftrightarrow \text { a unique solution } \widetilde{\boldsymbol{x}}=\boldsymbol{A}^{-1} \boldsymbol{d} \text { exists. }
\end{aligned}
$$

Thus the value of the determinant of $\boldsymbol{A}$ provides a convenient criterion for testing the nonsingularity of matrix $\boldsymbol{A}$ and the existence of a unique solution to the equation system $\boldsymbol{A x}=\boldsymbol{d}$.

## Rank of a Matrix Redefined

The rank of a matrix $\boldsymbol{A}$ was earlier defined to be the maximum number of linearly independent rows in $\boldsymbol{A}$. In view of the link between row independence and the nonvanishing of the determinant, we can redefine the
rank of an $m \times n$ matrix as the maximum order of a nonvanishing determinant that can be constructed from the rows and columns of that matrix. The rank of any matrix is a unique number.

Obviously, the rank can at most be $m$ or $n$ for a $m \times n$ matrix $\boldsymbol{A}$, whichever is smaller, because a determinant is defined only for a square matrix. Symbolically, this fact can be expressed as follows:

$$
\gamma(\boldsymbol{A}) \leq \min \{m, n\}
$$

## Example 5.3.8

$$
\gamma\left[\begin{array}{ccc}
1 & 3 & 2 \\
2 & 6 & 4 \\
-5 & 7 & 1
\end{array}\right]=2
$$

since $\left|\begin{array}{ccc}1 & 3 & 2 \\ 2 & 6 & 4 \\ -5 & 7 & 1\end{array}\right|=0$ and $\left|\begin{array}{ll}6 & 4 \\ 7 & 1\end{array}\right| \neq 0$.
One can also see this because the first two rows are linearly dependent, but the last two are independent, therefore the maximum number of linearly independent rows is equal to 2 .

## Properties of the rank:

1) The column rank and the row rank of a matrix are equal.
2) $\operatorname{rank}(\boldsymbol{A B}) \leq \min \{\operatorname{rank}(\boldsymbol{A}) ; \operatorname{rank}(\boldsymbol{B})\}$.
3) $\operatorname{rank}(\boldsymbol{A})=\operatorname{rank}\left(\boldsymbol{A} \boldsymbol{A}^{\prime}\right)=\operatorname{rank}\left(\boldsymbol{A}^{\prime} \boldsymbol{A}\right)$.

### 5.4 Finding the Inverse Matrix

If the matrix $\boldsymbol{A}$ in a linear-equation system $\boldsymbol{A} \boldsymbol{x}=\boldsymbol{d}$ is nonsingular, then $\boldsymbol{A}^{-1}$ exists, and the unique solution of the system will be $\overline{\boldsymbol{x}}=\boldsymbol{A}^{-1} \boldsymbol{d}$. We
have learned to test the nonsingularity of $\boldsymbol{A}$ by the criterion $|\boldsymbol{A}| \neq 0$. The next question is how we can find the inverse $\boldsymbol{A}^{-1}$ if $\boldsymbol{A}$ does pass that test.

## Expansion of a Determinant by Alien Cofactors

We have known that the value of a determinant $|\boldsymbol{A}|$ of order $n$ can be found by the Laplace expansion of any row or any column as follows;

$$
\begin{aligned}
|\boldsymbol{A}| & =\sum_{j=1}^{n} a_{i j}\left|\boldsymbol{C}_{i j}\right| \quad[\text { expansion by the } i \text { th row }] \\
& =\sum_{i=1}^{n} a_{i j}\left|\boldsymbol{C}_{i j}\right| \quad[\text { expansion by the } j \text { th column }]
\end{aligned}
$$

Now what happens if we replace one row (or column) by another row (or column), i.e., $a_{i j}$ by $a_{i^{\prime} j}$ for $i \neq i^{\prime}$ or by $a_{i j^{\prime}}$ for $j \neq j^{\prime}$. Then we have the following important property of determinants.

Property VIII. The expansion of a determinant by alien cofactors (the cofactors of a "wrong" row or column) always yields a value of zero. That is, we have
$\sum_{j=1}^{n} a_{i^{\prime} j}\left|\boldsymbol{C}_{i j}\right|=0 \quad\left(i \neq i^{\prime}\right)\left[\right.$ expansion by the $i^{\prime}$ th row and use of cofactors of $i$ th row $]$ $\sum_{j=1}^{n} a_{i j^{\prime}}\left|\boldsymbol{C}_{i j}\right|=0\left(j \neq j^{\prime}\right)$ [expansion by the $j^{\prime}$ th column and use of cofactors of $j$ th column $]$

The reason for this outcome lies in the fact that the above formula can be considered as the result of the regular expansion of a matrix that has two identical rows or columns.

Example 5.4.1 For the determinant

$$
|\boldsymbol{A}|=\left|\begin{array}{ccc}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{array}\right|,
$$

consider another determinant

$$
\left|\boldsymbol{A}^{*}\right|=\left|\begin{array}{lll}
a_{11} & a_{12} & a_{13} \\
a_{11} & a_{12} & a_{13} \\
a_{31} & a_{32} & a_{33}
\end{array}\right| .
$$

If we expand $\left|\boldsymbol{A}^{*}\right|$ by the second row, then we have

$$
0=\left|\boldsymbol{A}^{*}\right|=a_{11}\left|\boldsymbol{C}_{21}\right|+a_{12}\left|\boldsymbol{C}_{22}\right|+a_{13}\left|\boldsymbol{C}_{23}\right|=\sum_{j=1}^{3} a_{1 j}\left|\boldsymbol{C}_{2 j}\right| .
$$

## Matrix Inversion

Property VIII provides a way of finding the inverse of a matrix. For a $n \times n$ matrix $A$ :

$$
\boldsymbol{A}=\left[\begin{array}{ccc}
a_{11} & a_{12} & \cdots a_{1 n} \\
a_{21} & a_{22} & \cdots \\
a_{2 n} \\
\cdots & \cdots & \cdots \\
a_{n 1} & a_{n 2} & \cdots \\
a_{n n}
\end{array}\right],
$$

since each element of $\boldsymbol{A}$ has a cofactor $\left|\boldsymbol{C}_{i j}\right|$, we can form a matrix of cofactors by replacing each element $a_{i j}$ with its cofactor $\left|\boldsymbol{C}_{i j}\right|$. Such a cofactor matrix $\boldsymbol{C}=\left[\left|\boldsymbol{C}_{i j}\right|\right]$ is also $n \times n$. For our present purpose, however, the transpose of $\boldsymbol{C}$ is of more interest. This transpose $\boldsymbol{C}^{\prime}$ is commonly referred to as the adjoint of $\boldsymbol{A}$ and is denoted by adj $\boldsymbol{A}$. That is,

$$
\boldsymbol{C}^{\prime} \equiv \operatorname{adj} \boldsymbol{A} \equiv\left[\begin{array}{cccc}
\left|\boldsymbol{C}_{11}\right| & \left|\boldsymbol{C}_{21}\right| & \cdots & \left|\boldsymbol{C}_{n 1}\right| \\
\left|\boldsymbol{C}_{12}\right| & \left|\boldsymbol{C}_{22}\right| & \cdots & \left|\boldsymbol{C}_{n 2}\right| \\
\cdots & \cdots & \cdots & \cdots \\
\left|\boldsymbol{C}_{1 n}\right| & \left|\boldsymbol{C}_{2 n}\right| & \cdots & \left|\boldsymbol{C}_{n n}\right|
\end{array}\right]
$$

By utilizing the formula for the Laplace expansion and Property VI, we have

$$
\begin{aligned}
\boldsymbol{A} \boldsymbol{C}^{\prime} & =\left[\begin{array}{ccc}
a_{11} & a_{12} & \cdots \\
a_{21} & a_{22} & \cdots \\
\cdots & a_{2 n} \\
\cdots & \cdots & \cdots \\
a_{n 1} & a_{n 2} & \cdots
\end{array}\right]\left[\begin{array}{cccc}
\left|\boldsymbol{C}_{n n}\right| & \left|\boldsymbol{C}_{21}\right| & \cdots & \left|\boldsymbol{C}_{n 1}\right| \\
\left|\boldsymbol{C}_{12}\right| & \left|\boldsymbol{C}_{22}\right| & \cdots & \left|\boldsymbol{C}_{n 2}\right| \\
\cdots & \cdots & \cdots & \cdots \\
\left|\boldsymbol{C}_{1 n}\right| & \left|\boldsymbol{C}_{2 n}\right| & \cdots & \left|\boldsymbol{C}_{n n}\right|
\end{array}\right] \\
& =\left[\begin{array}{cccc}
\sum_{j=1}^{n} a_{1 j}\left|\boldsymbol{C}_{1 j}\right| & \sum_{j=1}^{n} a_{1 j}\left|\boldsymbol{C}_{2 j}\right| & \cdots & \sum_{j=1}^{n} a_{1 j}\left|\boldsymbol{C}_{n j}\right| \\
\sum_{j=1}^{n} a_{2 j}\left|\boldsymbol{C}_{1 j}\right| & \sum_{j=1}^{n} a_{2 j}\left|\boldsymbol{C}_{2 j}\right| & \cdots & \sum_{j=1}^{n} a_{2 j}\left|\boldsymbol{C}_{n j}\right| \\
\cdots & \ldots & \cdots & \cdots \\
\sum_{j=1}^{n} a_{n j}\left|\boldsymbol{C}_{1 j}\right| & \sum_{j=1}^{n} a_{n j}\left|\boldsymbol{C}_{2 j}\right| & \cdots & \sum_{j=1}^{n} a_{n j}\left|\boldsymbol{C}_{n j}\right|
\end{array}\right] \\
& =\left[\begin{array}{cccc}
|\boldsymbol{A}| & 0 & \cdots & 0 \\
0 & |\boldsymbol{A}| & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & |\boldsymbol{A}|
\end{array}\right] \\
& =|\boldsymbol{A}| \boldsymbol{I}_{n} .
\end{aligned}
$$

Therefore, by the uniqueness of $\boldsymbol{A}^{-1}$ of $\boldsymbol{A}$, we know

$$
\boldsymbol{A}^{-1}=\frac{\boldsymbol{C}^{\prime}}{|\boldsymbol{A}|}=\frac{\operatorname{adj} \boldsymbol{A}}{|\boldsymbol{A}|}
$$

Now we have found a way to invert the matrix $\boldsymbol{A}$.
Remark 5.4.1 In summary, the general procedures for finding the inverse
of a square matrix $\boldsymbol{A}$ are:

1. Calculate the determinant $|\boldsymbol{A}|$. If $|\boldsymbol{A}|=0$, then $\boldsymbol{A}$ is singular and has no inverse.
2. Find the cofactor matrix $\boldsymbol{C}=\left[\boldsymbol{C}_{i j}\right]$ by calculating the cofactor $\boldsymbol{C}_{i j}$ for each element of $\boldsymbol{A}$.
3. Obtain the adjugate matrix $C^{T}$, which is the transpose of $\boldsymbol{C}$.
4. Compute the inverse of $\boldsymbol{A}$ by using the formula $\boldsymbol{A}^{-1}=\frac{1}{|\boldsymbol{A}|} \boldsymbol{C}^{T}$.
5. Verify the result by computing the product $\boldsymbol{A} \boldsymbol{A}^{-1}$, which should equal the identity matrix $I$.

In particular, for a $2 \times 2$ matrix $\boldsymbol{A}=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]$, the cofactor matrix is:

$$
\boldsymbol{C}=\left[\begin{array}{ll}
\left|C_{11}\right| & \left|C_{12}\right| \\
\left|C_{21}\right| & \left|C_{22}\right|
\end{array}\right]=\left[\begin{array}{cc}
d & -c \\
-b & a
\end{array}\right] .
$$

Its transpose is:

$$
C^{\prime}=\left[\begin{array}{cc}
d & -b \\
-c & a
\end{array}\right]
$$

Therefore, the inverse is given by

$$
\begin{aligned}
\boldsymbol{A}^{-1} & =\frac{\operatorname{adj} \boldsymbol{A}}{|\boldsymbol{A}|} \\
& =\frac{1}{a d-c b}\left[\begin{array}{cc}
d & -b \\
-c & a
\end{array}\right]
\end{aligned}
$$

which is a very useful formula.

Example 5.4.2 $\boldsymbol{A}=\left[\begin{array}{ll}3 & 2 \\ 1 & 0\end{array}\right]$.
The inverse of $\boldsymbol{A}$ is given by

$$
\boldsymbol{A}^{-1}=\frac{1}{-2}\left[\begin{array}{cc}
0 & -2 \\
-1 & 3
\end{array}\right]=\left[\begin{array}{cc}
0 & 1 \\
\frac{1}{2} & -\frac{3}{2}
\end{array}\right]
$$

Example 5.4.3 Find the inverse of $\boldsymbol{B}=\left[\begin{array}{ccc}4 & 1 & -1 \\ 0 & 3 & 2 \\ 3 & 0 & 7\end{array}\right]$.
Since $|\boldsymbol{B}|=99 \neq 0, \boldsymbol{B}^{-1}$ exists. The cofactor matrix is

$$
\begin{aligned}
\boldsymbol{C} & =\left[\begin{array}{lll}
\left|\boldsymbol{C}_{11}\right| & \left|\boldsymbol{C}_{12}\right| & \left|\boldsymbol{C}_{13}\right| \\
\left|\boldsymbol{C}_{21}\right| & \left|\boldsymbol{C}_{22}\right| & \left|\boldsymbol{C}_{23}\right| \\
\left|\boldsymbol{C}_{31}\right| & \left|\boldsymbol{C}_{32}\right| & \left|\boldsymbol{C}_{33}\right|
\end{array}\right] \\
& =\left[\begin{array}{lll}
(-1)^{1+1}\left|\boldsymbol{M}_{11}\right| & (-1)^{1+2}\left|\boldsymbol{M}_{12}\right| & (-1)^{1+3}\left|\boldsymbol{M}_{13}\right| \\
(-1)^{2+1}\left|\boldsymbol{M}_{21}\right| & (-1)^{2+2}\left|\boldsymbol{M}_{22}\right| & (-1)^{2+3}\left|\boldsymbol{M}_{23}\right| \\
(-1)^{3+1}\left|\boldsymbol{M}_{31}\right| & (-1)^{3+2}\left|\boldsymbol{M}_{32}\right| & (-1)^{3+3}\left|\boldsymbol{M}_{33}\right|
\end{array}\right] \\
& =\left[\begin{array}{lll}
\left\lvert\, \begin{array}{cc}
3 & 2 \\
0 & 7
\end{array}\right. & -\left|\begin{array}{cc}
0 & 2 \\
3 & 7
\end{array}\right| & \left|\begin{array}{cc}
0 & 3 \\
3 & 0
\end{array}\right| \\
-\left|\begin{array}{cc}
1 & -1 \\
0 & 7
\end{array}\right| & \left|\begin{array}{cc}
4 & -1 \\
3 & 7
\end{array}\right| & -\left|\begin{array}{cc}
4 & 1 \\
3 & 0
\end{array}\right| \\
\left|\begin{array}{cc}
1 & -1 \\
3 & 2
\end{array}\right| & -\left|\begin{array}{cc}
4 & -1 \\
0 & 2
\end{array}\right| & \left|\begin{array}{cc}
4 & 1 \\
0 & 3
\end{array}\right|
\end{array}\right] \\
& =\left[\begin{array}{ccc}
21 & 6 & -9 \\
-7 & 31 & 3 \\
5 & -8 & 12
\end{array}\right] .
\end{aligned}
$$

Then

$$
\operatorname{adj} \boldsymbol{B}=\boldsymbol{C}^{\prime}=\left[\begin{array}{ccc}
21 & -7 & 5 \\
6 & 31 & -8 \\
-9 & 3 & 12
\end{array}\right]
$$

Therefore, we have

$$
\boldsymbol{B}^{-1}=\frac{1}{99}\left[\begin{array}{ccc}
21 & -7 & 5 \\
6 & 31 & -8 \\
-9 & 3 & 12
\end{array}\right]
$$

Example 5.4.4 $\boldsymbol{A}=\left[\begin{array}{lll}2 & 4 & 5 \\ 0 & 3 & 0 \\ 1 & 0 & 1\end{array}\right]$.
We have $|\boldsymbol{A}|=-9$ and

$$
\boldsymbol{A}^{-1}=-\frac{1}{9}\left[\begin{array}{ccc}
3 & -4 & -15 \\
0 & -3 & 0 \\
-3 & 4 & 6
\end{array}\right]
$$

### 5.5 Cramer's Rule

The method of matrix inversion just discussed enables us to derive a convenient way of solving a linear equation system, known as Cramer's rule.

## Derivation of the Cramer's Rule

Given a linear-equation system $\boldsymbol{A} \boldsymbol{x}=\boldsymbol{d}$, the solution can be written as

$$
\overline{\boldsymbol{x}}=\boldsymbol{A}^{-1} \boldsymbol{d}=\frac{1}{|\boldsymbol{A}|}(\operatorname{adj} \boldsymbol{A}) \boldsymbol{d}
$$ provided $\boldsymbol{A}$ is nonsingular. Thus,

$$
\begin{aligned}
\bar{x} & =\frac{1}{|\boldsymbol{A}|}\left[\begin{array}{cccc}
\left|\boldsymbol{C}_{11}\right| & \left|\boldsymbol{C}_{21}\right| & \cdots & \left|\boldsymbol{C}_{n 1}\right| \\
\left|\boldsymbol{C}_{12}\right| & \left|\boldsymbol{C}_{22}\right| & \cdots & \left|\boldsymbol{C}_{n 2}\right| \\
\ldots & \cdots & \cdots & \cdots \\
\left|\boldsymbol{C}_{1 n}\right| & \left|\boldsymbol{C}_{2 n}\right| & \cdots & \left|\boldsymbol{C}_{n n}\right|
\end{array}\right]\left[\begin{array}{c}
d_{1} \\
d_{2} \\
\cdots \\
d_{n}
\end{array}\right] \\
& =\frac{1}{|\boldsymbol{A}|}\left[\begin{array}{c}
\sum_{i=1}^{n} d_{i}\left|\boldsymbol{C}_{i 1}\right| \\
\sum_{i=1}^{n} d_{i}\left|\boldsymbol{C}_{i 2}\right| \\
\cdots \\
\sum_{i=1}^{n} d_{i}\left|\boldsymbol{C}_{i n}\right|
\end{array}\right] .
\end{aligned}
$$

That is, the $\bar{x}_{j}$ is given by

$$
\begin{aligned}
\bar{x}_{j} & =\frac{1}{|\boldsymbol{A}|} \sum_{i=1}^{n} d_{i}\left|\boldsymbol{C}_{i j}\right| \\
& =\frac{1}{|\boldsymbol{A}|}\left[\begin{array}{llllll}
a_{11} & a_{12} & \cdots & d_{1} & \cdots & a_{1 n} \\
a_{21} & a_{22} & \cdots & d_{2} & \cdots & a_{2 n} \\
\cdots & \cdots & \cdots & \cdots & & \\
a_{n 1} & a_{n 2} & \cdots & d_{n} & \cdots & a_{n n}
\end{array}\right] \\
& =\frac{1}{|\boldsymbol{A}|}\left|\boldsymbol{A}_{j}\right|,
\end{aligned}
$$

where $\left|\boldsymbol{A}_{j}\right|$ is obtained by replacing the $j$ th column of $|\boldsymbol{A}|$ with constant terms $d_{1}, \cdots, d_{n}$. This result is the statement of Cramer's rule.

Example 5.5.1 Let us solve

$$
\left[\begin{array}{cc}
2 & 3 \\
4 & -1
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]=\left[\begin{array}{l}
12 \\
10
\end{array}\right]
$$

for $x_{1}, x_{2}$ using Cramer's rule. Since

$$
|\boldsymbol{A}|=-14,\left|\boldsymbol{A}_{1}\right|=-42,\left|\boldsymbol{A}_{2}\right|=-28
$$

we have

$$
x_{1}=\frac{-42}{-14}=3, x_{2}=\frac{-28}{-14}=2 .
$$

## Example 5.5.2

$$
\begin{aligned}
& 5 x_{1}+3 x_{2}=30 \\
& 6 x_{1}-2 x_{2}=8
\end{aligned}
$$

We then have

$$
\begin{aligned}
& |\boldsymbol{A}|=\left|\begin{array}{cc}
5 & 3 \\
6 & -2
\end{array}\right|=-28 ; \\
& \left|\boldsymbol{A}_{1}\right|=\left|\begin{array}{cc}
30 & 3 \\
8 & -2
\end{array}\right|=-84 ; \\
& \left|\boldsymbol{A}_{2}\right|=\left|\begin{array}{cc}
5 & 30 \\
6 & 8
\end{array}\right|=-140 .
\end{aligned}
$$

Therefore, by Cramer's rule, we have

$$
\bar{x}_{1}=\frac{\left|\boldsymbol{A}_{1}\right|}{|\boldsymbol{A}|}=\frac{-84}{-28}=3 \text { and } \bar{x}_{2}=\frac{\left|\boldsymbol{A}_{2}\right|}{|\boldsymbol{A}|}=\frac{-140}{-28}=5 .
$$

## Example 5.5.3

$$
\begin{aligned}
& x_{1}+x_{2}+x_{3}=0 \\
& 12 x_{1}+2 x_{2}-3 x_{3}=5 \\
& 3 x_{1}+4 x_{2}+x_{3}=-4
\end{aligned}
$$

In the form of matrix

$$
\left[\begin{array}{ccc}
1 & 1 & 1 \\
12 & 2 & -3 \\
3 & 4 & 1
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right]=\left[\begin{array}{c}
0 \\
5 \\
-4
\end{array}\right] .
$$

We have

$$
|\boldsymbol{A}|=35,\left|\boldsymbol{A}_{3}\right|=35, \text { and thus } x_{3}=1
$$

## Example 5.5.4

$$
\begin{aligned}
& 7 x_{1}-x_{2}-x_{3}=0 \\
& 10 x_{1}-2 x_{2}+x_{3}=8 \\
& 6 x_{1}+3 x_{2}-2 x_{3}=7
\end{aligned}
$$

We have

$$
|\boldsymbol{A}|=-61,\left|\boldsymbol{A}_{1}\right|=-61,\left|\boldsymbol{A}_{2}\right|=-183,\left|\boldsymbol{A}_{3}\right|=-244 .
$$

Thus

$$
\begin{aligned}
& \bar{x}_{1}=\frac{\left|\boldsymbol{A}_{1}\right|}{|\boldsymbol{A}|}=1, \\
& \bar{x}_{2}=\frac{\left|\boldsymbol{A}_{2}\right|}{|\boldsymbol{A}|}=3, \\
& \bar{x}_{3}=\frac{\left|\boldsymbol{A}_{3}\right|}{|\boldsymbol{A}|}=4 .
\end{aligned}
$$

## Note on Homogeneous Linear-Equation System

A linear-equation system $\boldsymbol{A x}=\boldsymbol{d}$ is said to be a homogeneous-equation system if $\boldsymbol{d}=\mathbf{0}$, i.e., if $\boldsymbol{A} \boldsymbol{x}=\mathbf{0}$. If $|\boldsymbol{A}| \neq 0, \bar{x}=\mathbf{0}$ is a unique solution of $\boldsymbol{A x}=\mathbf{0}$ since $\overline{\boldsymbol{x}}=\boldsymbol{A}^{-1} \mathbf{0}=\mathbf{0}$. This is a "trivial solution." Thus, the only way to get a nontrivial solution from the homogeneous-equation system
is to have $|\boldsymbol{A}|=0$, i.e., $\boldsymbol{A}$ is singular. In this case, Cramer's rule is not directly applicable. Of course, this does not mean that we cannot obtain solutions; it means only that the solution is not unique. In fact, it has an infinite number of solutions.

If $r(\boldsymbol{A})=k<n$, we can delete $n-k$ dependent equations from the homogeneous-equation system $\boldsymbol{A x}=\mathbf{0}$, and then apply Cramer's rule to any $k$ variables, say $\left(x_{1}, \ldots, x_{k}\right)$ whose coefficient matrix has a rank $k$ and constant term in equation $i$ is $-\left(a_{i, k+1} x_{k+1}+\ldots, a_{i n} x_{n}\right)$.

## Example 5.5.5

$$
\begin{aligned}
& a_{11} x_{1}+a_{12} x_{2}=0 \\
& a_{21} x_{2}+a_{22} x_{2}=0
\end{aligned}
$$

If $|\boldsymbol{A}|=0$, then its rows are linearly dependent. As a result, one of two equations is redundant. By deleting, say, the second equation, we end up with one equation with two variables. The solutions are

$$
\bar{x}_{1}=-\frac{a_{12}}{a_{11}} x_{2} \text { if } a_{11} \neq 0
$$

For a linear-equation system with $n$ variables and $m$ equations, we have the following proposition.

Proposition 5.5.1 A necessary and sufficient condition for the existence of solution for a linear-equation system $\boldsymbol{A}_{m \times n} \boldsymbol{x}=\boldsymbol{d}$ with $n$ variables and $m$ equations is that the rank of $\boldsymbol{A}$ and the rank of the added matrix $[\boldsymbol{A} ; \boldsymbol{d}]$ are the same, i.e.,

$$
r(\boldsymbol{A})=r([\boldsymbol{A} ; \boldsymbol{d}]) .
$$

## Overview on Solution Outcomes for a linear-Equation System with Any

 Number of Variables and EquationsFor a general linear-equation system $\boldsymbol{A} \boldsymbol{x}=\boldsymbol{d}$, our discussion can be summarized as in the following table.

| $\|\boldsymbol{A}\|$ | $\boldsymbol{d}$ | $\boldsymbol{d} \neq \mathbf{0}$ | $\boldsymbol{d}=0$ |
| :--- | :---: | :---: | :---: |
| $\|\boldsymbol{A}\| \neq 0$ |  |  |  |\(\left.\left.\quad \begin{array}{c}The solution is unique <br>

and \overline{\boldsymbol{x}} \neq \mathbf{0}\end{array}\right) ~ $$
\begin{array}{c}\text { The solution is unique } \\
\text { and } \boldsymbol{x}=\mathbf{0}\end{array}
$$\right]\)

Table 5.1: The summary of solution for linear-equation system $\boldsymbol{A x}=\boldsymbol{d}$

### 5.6 Application to Market and National-Income Models

## Market Model:

The two-commodity model described in chapter 3 can be written as follows:

$$
\begin{aligned}
c_{1} P_{1}+c_{2} P_{2} & =-c_{0} \\
\gamma_{1} P_{1}+\gamma_{2} P_{2} & =-\gamma_{0} .
\end{aligned}
$$

Thus

$$
\begin{gathered}
|\boldsymbol{A}|=\left|\begin{array}{ll}
c_{1} & c_{2} \\
\gamma_{1} & \gamma_{2}
\end{array}\right|=c_{1} \gamma_{2}-c_{2} \gamma_{1}, \\
\left|\boldsymbol{A}_{1}\right|=\left|\begin{array}{ll}
-c_{0} & c_{2} \\
-\gamma_{0} & \gamma_{2}
\end{array}\right|=c_{2} \gamma_{0}-c_{0} \gamma_{2},
\end{gathered}
$$

5.6. APPLICATION TOMARKET AND NATIONAL-INCOME MODELS69

$$
\left|\boldsymbol{A}_{2}\right|=\left|\begin{array}{ll}
c_{1} & -c_{0} \\
\gamma_{1} & -\gamma_{0}
\end{array}\right|=c_{0} \gamma_{1}-c_{1} \gamma_{0} .
$$

Thus the equilibrium is given by

$$
\bar{P}_{1}=\frac{\left|\boldsymbol{A}_{1}\right|}{|\boldsymbol{A}|}=\frac{c_{2} \gamma_{0}-c_{0} \gamma_{2}}{c_{1} \gamma_{2}-c_{2} \gamma_{1}}
$$

and

$$
\bar{P}_{2}=\frac{\left|\boldsymbol{A}_{2}\right|}{|\boldsymbol{A}|}=\frac{c_{0} \gamma_{1}-c_{1} \gamma_{0}}{c_{1} \gamma_{2}-c_{2} \gamma_{1}} .
$$

## General Market Equilibrium Model:

Consider a market for three goods. The demand and supply for each good are given by:

$$
\begin{aligned}
& \left\{\begin{array}{l}
D_{1}=5-2 P_{1}+P_{2}+P_{3}, \\
S_{1}=-4+3 P_{1}+2 P_{2}
\end{array}\right. \\
& \left\{\begin{array}{l}
D_{2}=6+2 P_{1}-3 P_{2}+P_{3}, \\
S_{2}=3+2 P_{2}
\end{array}\right. \\
& \left\{\begin{array}{l}
D_{3}=20+P_{1}+2 P_{2}-4 P_{3}, \\
S_{3}=3+P_{2}+3 P_{3},
\end{array}\right.
\end{aligned}
$$

where $P_{i}$ is the price of good $i ; i=1 ; 2 ; 3$.
The equilibrium conditions are: $D_{i}=S_{i} ; i=1 ; 2 ; 3$, resulting in the
following three equations:

$$
\left\{\begin{array}{l}
5 P_{1}+P_{2}-P_{3}=9 \\
-2 P_{1}+5 P_{2}-P_{3}=3 \\
-P_{1}-P_{2}+7 P_{3}=17
\end{array}\right.
$$

This system of linear equations can then be solved via Cramer's rule

$$
\begin{aligned}
& \bar{P}_{1}=\frac{\left|\boldsymbol{A}_{1}\right|}{|\boldsymbol{A}|}=\frac{356}{178}=2, \\
& \bar{P}_{2}=\frac{\left|\boldsymbol{A}_{2}\right|}{|\boldsymbol{A}|}=\frac{356}{178}=2, \\
& \bar{P}_{3}=\frac{\left|\boldsymbol{A}_{3}\right|}{|\boldsymbol{A}|}=\frac{534}{178}=3 .
\end{aligned}
$$

## National-Income Model

Consider the simple national-income model:

$$
\begin{gathered}
Y=C+I_{0}+G_{0} \\
C=a+b Y(a>0,0<b<1) .
\end{gathered}
$$

These can be rearranged into the form

$$
\begin{gathered}
Y-C=I_{0}+G_{0} \\
-b Y+C=a
\end{gathered}
$$

While we can solve $\bar{Y}$ and $\bar{C}$ by Cramer's rule, here we solve this model by inverting the coefficient matrix.

Since $\boldsymbol{A}=\left[\begin{array}{cc}1 & -1 \\ -b & 1\end{array}\right]$, then $\boldsymbol{A}^{-1}=\frac{1}{1-b}\left[\begin{array}{ll}1 & 1 \\ b & 1\end{array}\right]$.

Hence

$$
\begin{aligned}
{\left[\begin{array}{ll}
\bar{Y} & \bar{C}
\end{array}\right] } & =\frac{1}{1-b}\left[\begin{array}{ll}
1 & 1 \\
b & 1
\end{array}\right]\left[\begin{array}{c}
I_{0}+G_{0} \\
a
\end{array}\right] \\
& =\frac{1}{1-b}\left[\begin{array}{c}
I_{0}+G_{0}+a \\
b\left(I_{0}+G_{0}\right)+a
\end{array}\right]
\end{aligned}
$$

### 5.7 Quadratic Forms

## Quadratic Forms

Definition 5.7.1 A function $q$ of $n$ variables is called a quadratic form if it has the following expression:

$$
\begin{aligned}
q\left(u_{1}, u_{2}, \cdots, u_{n}\right) & =d_{11} u_{1}^{2}+2 d_{12} u_{1} u_{2}+\cdots+2 d_{1 n} u_{1} u_{n} \\
& +d_{22} u_{2}^{2}+2 d_{23} u_{2} u_{3}+\cdots+2 d_{2 n} u_{2} u_{n} \\
& \cdots \\
& +d_{n n} u_{n}^{2} .
\end{aligned}
$$

This is, it is a polynomial having only second-order terms (either the square of a variable or the product of two variables).

If we let $d_{j i}=d_{i j}, i<j$, then $q\left(u_{1}, u_{2}, \cdots, u_{n}\right)$ can be written as

$$
\begin{aligned}
q\left(u_{1}, u_{2}, \cdots, u_{n}\right) & =d_{11} u_{1}^{2}+d_{12} u_{1} u_{2}+\cdots+d_{1 n} u_{1} u_{n} \\
& +d_{12} u_{2} u_{1}+d_{22} u_{2}^{2}+\cdots+d_{2 n} u_{2} u_{n} \\
& \cdots \\
& +d_{n 1} u_{n} u_{1}+d_{n 2} u_{n} u_{2}+\cdots+d_{n n} u_{n}^{2} \\
& =\sum_{i=1}^{n} \sum_{j=1}^{n} d_{i j} u_{i} u_{j} \\
& =\boldsymbol{u}^{\prime} \boldsymbol{D} \boldsymbol{u}
\end{aligned}
$$

where

$$
\boldsymbol{D}=\left[\begin{array}{cccc}
d_{11} & d_{12} & \cdots & d_{1 n} \\
d_{21} & d_{22} & \cdots & d_{2 n} \\
\cdots & \cdots & \cdots & \cdots \\
d_{n 1} & d_{n 2} & \cdots & d_{n n}
\end{array}\right],
$$

is called a matrix of quadratic form. Since $d_{i j}=d_{j i}, D$ is an $n$ th-order symmetric square matrix.

Example 5.7.1 A quadratic form in two variables:

$$
q=d_{11} u_{1}^{2}+d_{12} u_{1} u_{2}+d_{22} u_{2}^{2}
$$

The symmetric matrix is

$$
\left[\begin{array}{cc}
d_{11} & d_{12} / 2 \\
d_{12} / 2 & d_{22}
\end{array}\right]
$$

Then we have $q=\boldsymbol{u}^{\prime} \boldsymbol{D} \boldsymbol{u}$.

## Positive and Negative Definiteness:

Definition 5.7.2 Given a matrix $A$ that represents a quadratic form $q\left(u_{1}, u_{2}, \cdots, u_{n}\right)=$ $u^{T} A u$ :
(a) $q$ is positive definite (PD) if $q(u)>0$ for all $u \neq 0$;
(b) $q$ is positive semidefinite (PSD) if $q(u) \geqq 0$ for all $u$;
(c) $q$ is negative definite (ND) if $q(u)<0$ for all $u \neq 0$;
(d) $q$ is negative semidefinite (NSD) if $q(u) \leqq 0$ for all $u$.
(e) $q$ is indefinite (ID) if it does not satisfy any of the above conditions.

We may also say that a matrix $D$, for instance, is positive definite if its corresponding quadratic form $q(u)=u^{T} D u$ is positive definite.

## Example 5.7.2

$$
q=u_{1}^{2}+u_{2}^{2}
$$

is positive definite (PD),

$$
q=\left(u_{1}+u_{2}\right)^{2}
$$

is positive semidefinite (PSD), and

$$
q=u_{1}^{2}-u_{2}^{2}
$$

is indefinite.

## Determinantal Test for Sign Definiteness:

We state without proof that for the quadratic form $q(\boldsymbol{u})=\boldsymbol{u}^{\prime} \boldsymbol{D} \boldsymbol{u}$, the necessary and sufficient condition for positive definiteness is the order of the leading principal minors of $|\boldsymbol{D}|$, namely,

$$
\begin{gathered}
\left|\boldsymbol{D}_{1}\right|=d_{11}>0, \\
\left|\boldsymbol{D}_{2}\right|=\left|\begin{array}{ll}
d_{11} & d_{12} \\
d_{21} & d_{22}
\end{array}\right|>0,
\end{gathered}
$$

$$
\left|\boldsymbol{D}_{n}\right|=\left|\begin{array}{llll}
d_{11} & d_{12} & \cdots & d_{1 n} \\
d_{21} & d_{22} & \cdots & d_{2 n} \\
\cdots & \cdots & \cdots & \cdots \\
d_{n 1} & d_{n 2} & \cdots & d_{n n}
\end{array}\right|>0
$$

The corresponding necessary and sufficient condition for negative definiteness is that the order of the leading principal minors alternate in sign as follows:

$$
\left|\boldsymbol{D}_{1}\right|<0,\left|\boldsymbol{D}_{2}\right|>0,\left|\boldsymbol{D}_{3}\right|<0, \text { etc. }
$$

## Two-Variable Quadratic Form

Example 5.7.3 Is $q=5 u^{2}+3 u v+2 v^{2}$ either positive or negative definite? The symmetric matrix is

$$
\left[\begin{array}{cc}
5 & 1.5 \\
1.5 & 2
\end{array}\right] .
$$

Since the order of the leading principal minors of $|D|$ is $\left|D_{1}\right|=5$ and

$$
\left|D_{2}\right|=\left|\begin{array}{cc}
5 & 1.5 \\
1.5 & 2
\end{array}\right|=10-2.25=7.75>0
$$

so $q$ is positive definite.

## Three-Variable Quadratic Form

Example 5.7.4 Determine whether

$$
q=u_{1}^{2}+6 u_{2}^{2}+3 u_{3}^{2}-2 u_{1} u_{2}-4 u_{2} u_{3}
$$

is positive or negative definite. The matrix $\boldsymbol{D}$ corresponding this quadratic form is

$$
\boldsymbol{D}=\left[\begin{array}{ccc}
1 & -1 & 0 \\
-1 & 6 & -2 \\
0 & -2 & 3
\end{array}\right]
$$

and the order of the leading principal minors of $|\boldsymbol{D}|$ are

$$
\begin{gathered}
\left|\boldsymbol{D}_{1}\right|=1>0 \\
\left|\boldsymbol{D}_{2}\right|=\left|\begin{array}{cc}
1 & -1 \\
-1 & 6
\end{array}\right|=6-1=5>0
\end{gathered}
$$

and

$$
\left|\boldsymbol{D}_{3}\right|=\left|\begin{array}{ccc}
1 & -1 & 0 \\
-1 & 6 & -2 \\
0 & -2 & 3
\end{array}\right|=11>0
$$

Thus, the quadratic form is positive definite.
Example 5.7.5 Determine whether

$$
q=-3 u_{1}^{2}-3 u_{2}^{2}-5 u_{3}^{2}-2 u_{1} u_{2}
$$

is positive or negative definite. The matrix $\boldsymbol{D}$ corresponding this quadratic form is

$$
\boldsymbol{D}=\left[\begin{array}{ccc}
-3 & -1 & 0 \\
-1 & -3 & 0 \\
0 & 0 & -5
\end{array}\right]
$$

Leading the order of the leading principal minors of $D$ are

$$
\begin{gathered}
\left|\boldsymbol{D}_{1}\right|=-3<0, \\
\left|\boldsymbol{D}_{2}\right|=8>0
\end{gathered}
$$

$$
\left|\boldsymbol{D}_{3}\right|=-40<0 .
$$

Therefore, the quadratic form is negative definite.

### 5.8 Eigenvalues and Eigenvectors

Consider the matrix equation:

$$
\mathbf{D} \mathbf{x}=\lambda \mathbf{x}
$$

Any number $\lambda$ such that the equation $\mathbf{D x}=\lambda \mathbf{x}$ has a non-zero vectorsolution x is called an eigenvalue (also known as a characteristic root) of the matrix $\mathbf{D}$. Any non-zero vector x satisfying the above equation is called an eigenvector (also known as a characteristic vector) of $\mathbf{D}$ for the eigenvalue $\lambda$.

Recipe - How to calculate eigenvalues:
From $\boldsymbol{D} \boldsymbol{x}=\lambda \boldsymbol{x}$, we have the following homogeneous-equation system:

$$
(\boldsymbol{D}-\lambda \boldsymbol{I}) \boldsymbol{x}=\mathbf{0}
$$

Since we require that $x$ be non-zero, the determinant of $(\boldsymbol{D}-\lambda \boldsymbol{I})$ should vanish. Therefore all eigenvalues can be calculated as roots of the equation (which is often called the characteristic equation or the characteristic polynomial of $\boldsymbol{D}$ )

$$
|\boldsymbol{D}-\lambda \boldsymbol{I}|=0 .
$$

Example 5.8.1 Let

$$
\boldsymbol{D}=\left[\begin{array}{ccc}
3 & -1 & 0 \\
-1 & 3 & 0 \\
0 & 0 & 5
\end{array}\right]
$$

$$
\begin{aligned}
|\boldsymbol{D}-\lambda \boldsymbol{I}| & =\left|\begin{array}{ccc}
3-\lambda & -1 & 0 \\
-1 & 3-\lambda & 0 \\
0 & 0 & 5-\lambda
\end{array}\right| \\
& =(3-\lambda)(3-\lambda)(5-\lambda)-(5-\lambda) \\
& =(5-\lambda)(\lambda-2)(\lambda-4)=0,
\end{aligned}
$$

and therefore the eigenvalues are $\lambda_{1}=2, \lambda_{2}=4$, and $\lambda_{3}=5$.
For $\lambda_{1}=2$, we solve

$$
\left[\begin{array}{ccc}
3-2 & -1 & 2 \\
-1 & 3-2 & 0 \\
0 & 0 & 5-2
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right]
$$

Thus, the eigenvector, corresponding to $\lambda_{1}=2$, is $\boldsymbol{v}_{1}=c_{1}(1,1,0)^{\prime}$, where $c_{1}$ is an arbitrary real constant. Similarly, for $\lambda_{2}=4$ and $\lambda_{3}=5$, we have $\boldsymbol{v}_{2}=c_{2}(1,-1,0)^{\prime}$ and $\boldsymbol{v}_{3}=c_{3}(1,2,0)^{\prime}$, respectively.

## Properties of Eigenvalues:

Theorem 5.8.1 $A$ symmetric matrix $A$ is:
(1) Positive definite if and only if all its eigenvalues $\lambda_{i}$ are positive for $i=$ $1,2, \cdots, n$.
(2) Negative definite if and only if all its eigenvalues $\lambda_{i}$ are negative for $i=$ $1,2, \cdots, n$.
(3) Positive semi-definite if and only if all its eigenvalues $\lambda_{i}$ are non-negative for $i=1,2, \cdots, n$.
(4) Negative semi-definite if and only if all its eigenvalues $\lambda_{i}$ are non-positive for $i=1,2, \cdots, n$.
(5) Indefinite if it has at least one positive eigenvalue and at least one negative eigenvalue.

For a symmetric matrix $A$, there exists a convenient decomposition method. Matrix $A$ is said to be diagonalizable if there exist a non-singular matrix $P$ and a diagonal matrix $D$ such that:

$$
P^{-1} A P=D
$$

A unitary matrix $U$ is a complex square matrix such that its conjugate transpose $U^{*}$ (also denoted by $U^{\prime}$ ) is equal to its inverse, i.e., $U U^{*}=U^{*} U=$ $I$. For real matrices, a unitary matrix is the same as an orthogonal matrix $U$ where $U^{\prime}=U^{-1}$. "Orthogonal" implies that for any column vector $u$ of the matrix $U, u^{\prime} u=1$.

Theorem 5.8.2 (The Spectral Theorem for Symmetric Matrices) Suppose that $A$ is a real symmetric matrix of order $n$, and let $\lambda_{1}, \ldots, \lambda_{n}$ be its eigenvalues. Then there exists an orthogonal matrix $U$ such that:

$$
U^{-1} A U=\left[\begin{array}{lll}
\lambda_{1} & & 0 \\
& \ddots & \\
0 & & \lambda_{n}
\end{array}\right]
$$

or equivalently:

$$
A=U\left[\begin{array}{lll}
\lambda_{1} & & 0 \\
& \ddots & \\
0 & & \lambda_{n}
\end{array}\right] U^{\prime}
$$

Usually, $U$ is the orthogonal matrix formed by eigenvectors. It has the property $U^{\prime} U=U U^{\prime}=I$.

Example 5.8.2 Diagonalize the matrix

$$
\boldsymbol{A}=\left[\begin{array}{ll}
1 & 2 \\
2 & 4
\end{array}\right]
$$

First, we need to find the eigenvalues:

$$
\left|\begin{array}{cc}
1-\lambda & 2 \\
2 & 4-\lambda
\end{array}\right|=\lambda(\lambda-5)=0
$$

i.e., $\lambda_{1}=0$ and $\lambda_{2}=5$.

For $\lambda_{1}=0$, we solve

$$
\left[\begin{array}{cc}
1-0 & 2 \\
2 & 4-0
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right] .
$$

The eigenvector, corresponding to $\lambda_{1}=0$, is $\boldsymbol{v}_{1}=c_{1}(2,-1)^{\prime}$, where $C_{1}$ is an arbitrary real constant. Similarly, for $\lambda_{2}=5$, we have $\boldsymbol{v}_{2}=c_{2}(1,2)^{\prime}$.

Let us normalize the eigenvectors, i.e. let us pick constants $C_{i}$ such that $\boldsymbol{v}_{i}^{\prime} \boldsymbol{v}_{i}=1$. We get

$$
\boldsymbol{v}_{1}=\left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right), \boldsymbol{v}_{2}=\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) .
$$

Thus the diagonalization matrix $\boldsymbol{U}$ is

$$
\boldsymbol{U}=\left[\begin{array}{cc}
\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\
\frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}}
\end{array}\right]
$$

You can easily check that

$$
\boldsymbol{U}^{-1} \boldsymbol{A} \boldsymbol{U}=\left[\begin{array}{ll}
0 & 0 \\
0 & 5
\end{array}\right]
$$

The trace of a square matrix of order $n$ is the sum of the $n$ elements on its principal diagonal, i.e., $\operatorname{tr}(\boldsymbol{A})=\sum_{i=1}^{n} a_{i i}$.

## Properties of the Trace:

(1) $\operatorname{tr}(A)=\lambda_{1}+\cdots+\lambda_{n}$;
(2) If $A$ and $B$ have the same dimension, then $\operatorname{tr}(A+B)=$ $\operatorname{tr}(A)+\operatorname{tr}(B) ;$
(3) If $a$ is a real number, $\operatorname{tr}(a A)=a \cdot \operatorname{tr}(A)$;
(4) $\operatorname{tr}(A B)=\operatorname{tr}(B A)$, if $A B$ is a square matrix;
(5) $\operatorname{tr}\left(A^{\prime}\right)=\operatorname{tr}(A)$;
(6) $\operatorname{tr}\left(A^{\prime} A\right)=\sum_{i=1}^{n} \sum_{j=1}^{n} a_{i j}^{2}$.

Note that in property (1), the eigenvalues $\lambda_{1}, \cdots, \lambda_{n}$ referred to are the eigenvalues of $A$.

### 5.9 Vector Spaces

A (real) vector space is a nonempty set $V$ of objects together with an additive operation $+: V \times V \rightarrow V,+(\boldsymbol{u}, \boldsymbol{v})=\boldsymbol{u}+\boldsymbol{v}$ and a scalar multiplicative operation $\cdot: \mathbb{R} \times V \rightarrow V, \cdot(a, \boldsymbol{u})=a \boldsymbol{u}$ which satisfies the following axioms for any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$ and any $a, b \in \mathbb{R}$ where $\mathbb{R}$ is the set of all real numbers:

1. $(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}=\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$;
2. $\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}$;
3. $\mathbf{0}+\boldsymbol{u}=\boldsymbol{u}$;
4. $\boldsymbol{u}+(-\boldsymbol{u})=\mathbf{0}$;
5. $a(\boldsymbol{u}+\boldsymbol{v})=a \boldsymbol{u}+a \boldsymbol{v}$;
6. $(a+b) \boldsymbol{u}=a \boldsymbol{u}+b \boldsymbol{u}$;
7. $a(b \boldsymbol{u})=(a b) \boldsymbol{u}$;
8. $1 \boldsymbol{u}=\boldsymbol{u}$.

The objects of a vector space $V$ are called the vectors, the operations + and • are called the vector addition and scalar multiplication, respectively. The element $\mathbf{0} \in V$ is the zero vector and $-\boldsymbol{v}$ is the additive inverse of $V$.

Example 5.9.1 (The $\boldsymbol{n}$-Dimensional Vector Space $\mathbb{R}^{n}$ ) For $\mathbb{R}^{n}$, consider $\boldsymbol{u}, \boldsymbol{v} \in$ $\mathbb{R}^{n}, \boldsymbol{u}=\left(u_{1}, u_{2}, \cdots, u_{n}\right)^{\prime}, \boldsymbol{v}=\left(v_{1}, v_{2}, \cdots, v_{n}\right)^{\prime}$ and $a \in \mathbb{R}$. Define the additive operation and the scalar multiplication as follows:

$$
\begin{gathered}
\boldsymbol{u}+\boldsymbol{v}=\left(u_{1}+v_{1}, \cdots, u_{n}+v_{n}\right)^{\prime}, \\
a \boldsymbol{u}=\left(a u_{1}, \cdots, a u_{n}\right)^{\prime} .
\end{gathered}
$$

It is not difficult to verify that $\mathbb{R}^{n}$ together with these operations is a vector space.

Let $V$ be a vector space. An inner product or scalar product in $V$ is a function $s: V \times V \rightarrow \mathbb{R}, s(\boldsymbol{u}, \boldsymbol{v})=\boldsymbol{u} \cdot \boldsymbol{v}$ which satisfies the following properties:

1. $\boldsymbol{u} \cdot \boldsymbol{v}=\boldsymbol{v} \cdot \boldsymbol{u}$,
2. $\boldsymbol{u} \cdot(\boldsymbol{v}+\boldsymbol{w})=\boldsymbol{u} \cdot \boldsymbol{v}+\boldsymbol{u} \cdot \boldsymbol{w}$,
3. $a(\boldsymbol{u} \cdot \boldsymbol{v})=(a \boldsymbol{u}) \cdot \boldsymbol{v}=\boldsymbol{u} \cdot(a \boldsymbol{v})$,
4. $\boldsymbol{u} \cdot \boldsymbol{u} \geq 0$ and $\boldsymbol{u} \cdot \boldsymbol{u}=0$ iff $\boldsymbol{u}=\mathbf{0}$.

Example 5.9.2 Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^{n}, \boldsymbol{u}=\left(u_{1}, u_{2}, \cdots, u_{n}\right)^{\prime}, \boldsymbol{v}=\left(v_{1}, v_{2}, \cdots, v_{n}\right)^{\prime}$. Then $\boldsymbol{u} \cdot \boldsymbol{v}=u_{1} v_{1}+\cdots+u_{n} v_{n}$.

Let $V$ be a vector space and $\boldsymbol{v} \in V$. The norm of magnitude is a function $\|\cdot\|: V \rightarrow \mathbb{R}$ defined as $\|\boldsymbol{v}\|=\sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}$. For any $\boldsymbol{v} \in V$ and any $a \in \mathbb{R}$, we have the following properties:

1. $\|a \boldsymbol{u}\|=|a|\|\boldsymbol{u}\|$;
2. $\|\boldsymbol{u}+\boldsymbol{v}\| \leq\|\boldsymbol{u}\|+\|\boldsymbol{v}\|$;
3. $|\boldsymbol{u} \cdot \boldsymbol{v}| \leq\|\boldsymbol{u}\| \times\|\boldsymbol{v}\|$.

The nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel if there exists $a \in \mathbb{R}$ such that $\boldsymbol{u}=a \boldsymbol{v}$.

The vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal or perpendicular if their scalar product is zero, that is, if $\boldsymbol{u} \cdot \boldsymbol{v}=0$.

The angle between vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\arccos \left(\frac{u v}{\|u\|\|v\|}\right)$
A nonempty subset $S$ of a vector space $V$ is a subspace of $V$ if for any $\boldsymbol{u}, \boldsymbol{v} \in S$ and $a \in \mathbb{R}$

$$
\boldsymbol{u}+\boldsymbol{v} \in S \text { and } a \boldsymbol{u} \in S
$$

Example 5.9.3 $V$ is a subset of itself. $\{0\}$ is also a subset of $V$. These subspaces are called proper subspaces.

Example 5.9.4 $L=\{(x, y) \mid y=m x+n\}$ where $m, n \in \mathbb{R}$ and $m \neq 0$ is a subspace of $\mathbb{R}^{2}$.

Let $\boldsymbol{u}_{1}, \boldsymbol{u}_{2}, \cdots, \boldsymbol{u}_{k}$ be vectors in a vector space $V$. The set $S$ of all linear combinations of these vectors

$$
S=\left\{a_{1} \boldsymbol{u}_{1}+a_{2} \boldsymbol{u}_{2}+\cdots, a_{k} \boldsymbol{u}_{k} \mid a_{1}, \cdots, a_{k} \in \mathbb{R}\right\}
$$

is called the subspace generated or spanned by the vectors $\boldsymbol{u}_{1}, \boldsymbol{u}_{2}, \cdots, \boldsymbol{u}_{k}$ and denoted as $\operatorname{sp}\left(\boldsymbol{u}_{1}, \boldsymbol{u}_{2}, \cdots, \boldsymbol{u}_{k}\right)$. One can prove that $S$ is a subspace of $V$.

Example 5.9.5 Let $\boldsymbol{u}_{1}=(2,-1,1)^{\prime}, \boldsymbol{u}_{2}=(3,4,0)^{\prime}$. Then the subspace of $\mathbb{R}^{3}$ generated by $\boldsymbol{u}_{1}$ and $\boldsymbol{u}_{2}$ is

$$
s p\left(\boldsymbol{u}_{1}, \boldsymbol{u}_{2}\right)=\left\{(2 a+3 b,-a+4 b, a)^{\prime} \mid a, b \in \mathbb{R}\right\} .
$$

As we discussed in Chapter 4, a set of vectors $\left\{\boldsymbol{u}_{1}, \boldsymbol{u}_{2}, \cdots, \boldsymbol{u}_{k}\right\}$ in a vector space $V$ is linearly dependent if there exists the real numbers $a_{1}, a_{2}, \cdots, a_{k}$, not all zero, such that $a_{1} \boldsymbol{u}_{1}+a_{2} \boldsymbol{u}_{2}+\cdots+a_{k} \boldsymbol{u}_{k}=\mathbf{0}$. In other words, the set of vectors in a vector space is linearly dependent if and only if one vector can be written as a linear combination of the others. A set of vectors $\left\{\boldsymbol{u}_{1}, \boldsymbol{u}_{2}, \cdots, \boldsymbol{u}_{k}\right\}$ in a vector space $V$ is linearly independent if it is not linearly dependent.

Properties: Let $\left\{\boldsymbol{u}_{1}, \boldsymbol{u}_{2}, \cdots, \boldsymbol{u}_{k}\right\}$ be $n$ vectors in $\mathbb{R}^{n}$. The following conditions are equivalent:
i) The vectors are independent.
ii) The matrix having these vectors as columns is nonsingular.
iii) The vectors generate $\mathbb{R}^{n}$.

A set of vectors $\left\{\boldsymbol{u}_{1}, \boldsymbol{u}_{2}, \cdots, \boldsymbol{u}_{k}\right\}$ in $V$ is a basis for $V$ if it, first, generates $V$, and, second, is linearly independent.

Example 5.9.6 Consider the following vectors in $\mathbb{R}^{n}$. $e_{i}=(0, \cdots, 0,1,0, \cdots, 0)^{\prime}$, where 1 is in the $i$ th position, $i=1, \cdots, n$. The set $E_{n}=\left\{e_{1}, e_{2}, \cdots, e_{n}\right\}$ forms a basis for $\mathbb{R}^{n}$ which is called the standard basis.

Let $V$ be a vector space and $\boldsymbol{B}=\left\{\boldsymbol{u}_{1}, \boldsymbol{u}_{2}, \cdots, \boldsymbol{u}_{k}\right\}$ a basis for $V$. Since $\boldsymbol{B}$ generates $V$, for any $\boldsymbol{u} \in V$, there exists the real numbers $x_{1}, x_{2}, \cdots, x_{n}$ such that $\boldsymbol{u}=x_{1} \boldsymbol{u}_{1}+\cdots+x_{n} \boldsymbol{u}_{n}$. The column vector $\boldsymbol{x}=\left(x_{1}, x_{2}, \cdots, x_{n}\right)^{\prime}$ is called the vector of coordinates of $\boldsymbol{u}$ with respect to $\boldsymbol{B}$.

Example 5.9.7 Consider the vector space $\mathbb{R}^{n}$ with the standard basis $E_{n}$. For any $\boldsymbol{u}=\left(u_{1}, \cdots, u_{n}\right)^{\prime}$, we can represent $\boldsymbol{u}$ as $\boldsymbol{u}=u_{1} e_{1}+\cdots+u_{n} e_{n}$; therefore, $\left(u_{1}, \cdots, u_{n}\right)^{\prime}$ is the vector of coordinates of $\boldsymbol{u}$ with respect to $E_{n}$.

Example 5.9.8 Consider the vector space $\mathbb{R}^{2}$. Let us find the coordinate vector of $(-1,2)^{\prime}$ with respect to the basis $\boldsymbol{B}=(1,1)^{\prime},(2,-3)^{\prime}$ (i.e., find $\left.(-1,2)_{B}^{\prime}\right)$. We have to solve for $a, b$ such that $(-1,2)^{\prime}=a(1,1)^{\prime}+b(2,-3)^{\prime}$. Solving the system $a+2 b=-1$ and $a-3 b=2$, we find $a=\frac{1}{5}$ and $b=-\frac{3}{5}$. Thus, $(-1,2)_{B}^{\prime}=\left(\frac{1}{5},-\frac{3}{5}\right)^{\prime}$.

The dimension of a vector space $V \operatorname{dim}(V)$ is the number of elements in any basis for $V$.

Example 5.9.9 The dimension of the vector space $\mathbb{R}^{n}$ with the standard basis $E_{n}$ is $n$.

Let $U$ and $V$ be two vector spaces. A linear transformation of $U$ into $V$ is a mapping $T: U \rightarrow V$ such that for any $\boldsymbol{u}, \boldsymbol{v} \in U$ and any $a, b \in \mathbb{R}$, we have

$$
T(a \boldsymbol{u}+b \boldsymbol{v})=a T(\boldsymbol{u})+b T(\boldsymbol{v}) .
$$

Example 5.9.10 Let $\boldsymbol{A}$ be a $m \times n$ real matrix. The mapping $T: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ defined by $T(\boldsymbol{u})=\boldsymbol{A} \boldsymbol{u}$ is a linear transformation.

## Properties:

Let $U$ and $V$ be two vector spaces, $\boldsymbol{B}=\left(b_{1}, \cdots, b_{n}\right)$ a basis for $U$ and $\boldsymbol{C}=\left(c_{1}, \cdots, c_{m}\right)$ a basis for $V$.

1. Any linear transformation $T$ can be represented by an $m \times$ $n$ matrix $\boldsymbol{A}_{T}$ whose $i$ th column is the coordinate vector of $T\left(b_{i}\right)$ relative to $\boldsymbol{C}$.
2. If $x=\left(x_{1}, \cdots, x_{n}\right)^{\prime}$ is the coordinate vector of $\boldsymbol{u} \in U$ relative to $\boldsymbol{B}$ and $y=\left(y_{1}, \cdots, y_{m}\right)^{\prime}$ is the coordinate vector of $T(\boldsymbol{u})$ relative to $\boldsymbol{C}$, then $T$ defines the following transformation of coordinates:

$$
\boldsymbol{y}=\boldsymbol{A}_{T} \boldsymbol{x} \text { for any } \boldsymbol{u} \in U .
$$

The matrix $\boldsymbol{A}_{T}$ is called the matrix representation of $T$ relative to bases $B$ and $\boldsymbol{C}$.

Remark 5.9.1 Any linear transformation is uniquely determined by a transformation of coordinates.

Example 5.9.11 Consider the linear transformation $T: \mathbb{R}^{3} \rightarrow \mathbb{R}^{2}, T\left((x, y, z)^{\prime}\right)=$ $(x-2 y, x+z)^{\prime}$ and bases $\boldsymbol{B}=\left\{(1,1,1)^{\prime},(1,1,0)^{\prime},(1,0,0)^{\prime}\right\}$ for $\mathbb{R}^{3}$ and $\boldsymbol{C}=\left\{(1,1)^{\prime},(1,0)^{\prime}\right\}$ for $\mathbb{R}^{2}$. How can we find the matrix representation of $T$ relative to bases $\boldsymbol{B}$ and $\boldsymbol{C}$ ?

We have

$$
T\left((1,1,1)^{\prime}\right)=(-1,2), T\left((1,1,0)^{\prime}\right)=(-1,1), T\left((1,0,0)^{\prime}\right)=(1,1) .
$$

The columns of $\boldsymbol{A}_{T}$ are formed by the coordinate vectors of $T\left((1,1,1)^{\prime}\right)$, $T\left((1,1,0)^{\prime}\right), T\left((1,0,0)^{\prime}\right)$ relative to $\boldsymbol{C}$. Applying the procedure developed in Example 5.9.8, we find

$$
\boldsymbol{A}_{T}=\left[\begin{array}{ccc}
2 & 1 & 1 \\
-3 & -2 & 0
\end{array}\right]
$$

Let $V$ be a vector space of dimension $n, \boldsymbol{B}$ and $\boldsymbol{C}$ be two bases for $V$, and $I: V \rightarrow V$ be the identity transformation $((I(v)=v$ for all $v \in V)$. The change-of-basis matrix $\boldsymbol{D}$ relative to $\boldsymbol{B}, \boldsymbol{C}$ is the matrix representation of $I$ to $B, C$.

Example 5.9.12 For $\boldsymbol{u} \in V$, let $\boldsymbol{x}=\left(x_{1}, \cdots, x_{n}\right)^{\prime}$ be the coordinate vector of $\boldsymbol{u}$ relative to $\boldsymbol{B}$ and $\boldsymbol{y}=\left(y_{1}, \cdots, y_{n}\right)^{\prime}$ is the coordinate vector of $\boldsymbol{u}$ relative to $\boldsymbol{C}$. If $\boldsymbol{D}$ is the change-of-basis matrix relative to $\boldsymbol{B}, \boldsymbol{C}$ then $\boldsymbol{y}=\boldsymbol{C x}$. The change-of-basis matrix relative to $\boldsymbol{C}, \boldsymbol{B}$ is $\boldsymbol{D}^{-1}$.

Example 5.9.13 Given the following bases for $\mathbb{R}^{2}: \boldsymbol{B}=\left\{(1,1)^{\prime},(1,0)^{\prime}\right\}$ and $\boldsymbol{C}=\left\{(0,1)^{\prime},(1,1)^{\prime}\right\}$, find the change-of-basis matrix $\boldsymbol{D}$ relative to $\boldsymbol{B}, \boldsymbol{C}$. The columns of $\boldsymbol{D}$ are the coordinate vectors of $(1,1)^{\prime},(1,0)^{\prime}$ relative to $\boldsymbol{C}$. Following Example 5.9.8, we find

$$
\boldsymbol{D}=\left[\begin{array}{cc}
0 & -1 \\
1 & 1
\end{array}\right]
$$

## Chapter 6

## Comparative Statics and the Concept of Derivative

### 6.1 The Nature of Comparative Statics

Comparative statics is concerned with comparing different equilibrium states that are associated with different sets of values of parameters and exogenous variables. When the value of some parameter or exogenous variable that is associated with an initial equilibrium changes, we will have a new equilibrium. The question posed in the comparative-static analysis is: How does the new equilibrium compare with the old?

It should be noted that in comparative-static analysis, we don't concern ourselves with the process of adjusting the variables; we merely compare the initial (pre-change) equilibrium state with the new (post-change) equilibrium state. We also preclude the possibility of instability of equilibrium because we assume that the equilibrium is attainable.

It should be clear that the problem under consideration is essentially one of finding a rate of change: the rate of change of the equilibrium value of an endogenous variable with respect to the change in a particular pa-
rameter or exogenous variable. For this reason, the mathematical concept of derivative takes on significant importance in comparative statics.

### 6.2 Rate of Change and the Derivative

We want to study the rate of change of any variable $y$ in response to a change in another variable $x$, where the two variables are related to each other by the function:

$$
y=f(x) .
$$

In the comparative statics context, the variable $y$ represents the equilibrium value of an endogenous variable, and $x$ represents some parameter.

## The Difference Quotient

We use the symbol $\Delta$ to denote the change from one point, say $x_{0}$, to another point, say $x_{1}$. Thus $\Delta x=x_{1}-x_{0}$. When $x$ changes from an initial value $x_{0}$ to a new value $x_{0}+\Delta x$, the value of the function $y=f(x)$ changes from $f\left(x_{0}\right)$ to $f\left(x_{0}+\Delta x\right)$. The change in $y$ per unit of change in $x$ can be represented by the difference quotient.

$$
\frac{\Delta y}{\Delta x}=\frac{f\left(x_{0}+\Delta x\right)-f\left(x_{0}\right)}{\Delta x} .
$$

Example 6.2.1 $y=f(x)=3 x^{2}-4$.
Then $f\left(x_{0}\right)=3 x_{0}^{2}-4, f\left(x_{0}+\Delta x\right)=3\left(x_{0}+\Delta x\right)^{2}-4$,
and thus,

$$
\begin{aligned}
\frac{\Delta y}{\Delta x} & =\frac{f\left(x_{0}+\Delta x\right)-f\left(x_{0}\right)}{\Delta x} \\
& =\frac{3\left(x_{0}+\Delta x\right)^{2}-4-\left(3 x_{0}^{2}-4\right)}{\Delta x} \\
& =\frac{6 x_{0} \Delta x+3(\Delta x)^{2}}{\Delta x} \\
& =6 x_{0}+3 \Delta x .
\end{aligned}
$$

## The Derivative

Frequently, we are interested in the rate of change of $y$ when $\Delta x$ is very small. In particular, we want to know the rate of $\Delta y / \Delta x$ when $\Delta x$ approaches to zero. If, as $\Delta x \rightarrow 0$, the limit of the difference quotient $\Delta y / \Delta x$ exits, that limit is called the derivative of the function $y=f(x)$, and the derivative is denoted by

$$
\frac{d y}{d x} \equiv y^{\prime} \equiv f^{\prime}(x) \equiv \lim _{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}
$$

Remark 6.2.1 It is important to note several points about the derivative:
(1) The derivative is a function of $x_{0}$ only, whereas the difference quotient is a function of both $x_{0}$ and $\Delta x$.
(2) The derivative is a limit of the difference quotient and, therefore, is a measure of some rate of change. As $\Delta x$ approaches zero, the rate measured by the derivative represents an instantaneous rate of change.

Example 6.2.2 Referring to the function $y=3 x^{2}-4$ again. Since

$$
\frac{\Delta y}{\Delta x}=6 x+3 \Delta x
$$

we have $\frac{d y}{d x}=6 x$.

### 6.3 The Derivative and the Slope of a Curve

In elementary economics, the marginal cost $M C$ of a total-cost function $C=f(Q)$, where $C$ is the total cost and $Q$ is the output, is defined as $M C=\Delta C / \Delta Q$, where $\Delta Q$ refers to an infinitesimal change. The marginal cost $M C$ is a continuous variable.

The slope of the total-cost curve is a well-known way to measure $M C$. However, the slope of the total-cost curve is just the limit of the ratio $\Delta C / \Delta Q$ as $\Delta Q \rightarrow 0$. The concept of the slope of a curve is therefore the geometric counterpart of the concept of the derivative.


Figure 6.1: Graphical illustrations of the slope of the total cost curve and the marginal cost.

### 6.4 The Concept of Limit

In the previous section, we defined the derivative of a function $y=f(x)$ as the limit of $\Delta y / \Delta x$ as $\Delta x \rightarrow 0$. We now turn our attention to the concept of limit.

Given a function $q=q(v)$, the concept of limit is concerned with the value that $q$ approaches as $v$ approaches a specific value, denoted by $N$. That is, we are interested in what happens to $\lim _{v \rightarrow N} q$ as $v$ approaches $N$, where $N$ can be any number, such as $N=0, N=+\infty$, or $N=-\infty$.

As $v$ approaches $N$, the variable $v$ can approach $N$ either from values greater than $N$ or from values less than $N$. If $q$ approaches a finite number $L$ as $v \rightarrow N$ from values less than $N$, we call $L$ the left-side limit of $q$. Similarly, we call $L$ the right-side limit of $q$ if $v \rightarrow N$ from values greater than $N$. The left-side limit and right-side limit of $q$ are denoted by $\lim _{v \rightarrow N^{-}} q$ and $\lim _{v \rightarrow N^{+}} q$, respectively.

The limit of $q$ at $N$ exists if and only if the left-side limit and right-side limit are equal. That is,

$$
\lim _{v \rightarrow N^{-}} q=\lim _{v \rightarrow N^{+}} q,
$$

and is denoted by $\lim _{v \rightarrow N} q=L$. Note that $L$ must be a finite number. If $\lim _{v \rightarrow N} q=\infty$ or $-\infty$, we say that $q$ has no limit or an infinite limit. It is important to realize that the symbol $\infty$ is not a number, and therefore cannot be subjected to the usual algebraic operations.

## Graphical Illustrations

There are several possible situations regrading the limit of a function, which are shown in the following diagrams.


Figure 6.2: Possible Situations regarding the limit of a function $q=g(v)$.

## Evaluation of a Limit

Let us now illustrate the algebraic evaluation of a limit of a given function $q=g(v)$.

Example 6.4.1 Given $q=2+v^{2}$, find $\lim _{v \rightarrow 0} q$. It is clear that $\lim _{v \rightarrow 0^{-}} q=2$ and $\lim _{v \rightarrow 0^{+}} q=2$ and $v^{2} \rightarrow 0$ as $v \rightarrow 0$. Thus $\lim _{v \rightarrow 0} q=2$.

Note that, in evaluating $\lim _{v \rightarrow N} q$, we only let $v$ approach $N$ but, as a rule, do not let $v=N$. Indeed, sometimes $N$ is not even in the domain of the function $q=g(v)$.

## Example 6.4.2 Consider

$$
q=\left(1-v^{2}\right) /(1-v) .
$$

For this function, $N=1$ is not in the domain of the function, and we cannot set $v=1$ since it would involve division by zero. Moreover, even the limit-evaluation procedure of letting $v$ approach 1 will cause difficulty since $(1-v) \rightarrow 0$ as $v \rightarrow 1$.

One way out of this difficulty is to try to transform the given ratio to a form in which $v$ will not appear in the denominator. Since

$$
q=\frac{1-v^{2}}{1-v}=\frac{(1-v)(1+v)}{1-v}=1+v(v \neq 1)
$$

and $v \rightarrow 1$ implies $v \neq 1$ and $(1+v) \rightarrow 2$ as $v \rightarrow 1$, we have $\lim _{v \rightarrow 1} q=2$.
Example 6.4.3 Find $\lim _{v \rightarrow \infty} \frac{2 v+5}{v+1}$.

$$
\text { Since } \frac{2 v+5}{v+1}=\frac{2(v+1)+3}{v+1}=2+\frac{3}{v+1} \text { and } \lim _{v \rightarrow \infty} \frac{3}{v+1}=0 \text {, so } \lim _{v \rightarrow \infty} \frac{2 v+5}{v+1}=2 \text {. }
$$

## Formal View of the Limit Concept

Definition 6.4.1 The number $L$ is said to be the limit of $q=g(v)$ as $v$ approaches $N$ if, for every neighborhood of $L$, there can be found a corresponding neighborhood of $N$ (excluding the point $v=N$ ) in the domain of the function such that, for every value of $v$ in that neighborhood, its image lies in the chosen $L$-neighborhood. Here a neighborhood of a point $L$ is an open interval defined by

$$
\left(L-a_{1}, L+a_{2}\right)=\left\{q \mid L-a_{1}<q<L+a_{2}\right\} \text { for } a_{1}>a_{2}>0
$$



Figure 6.3: The graphical representation of the limit defined in term of neighborhoods.

### 6.5 Inequality and Absolute Values

Rules of Inequalities:

## Transitivity:

$a>b$ and $b>c$ implies $a>c$;
$a \geq b$ and $b \geq c$ implies $a \geq c$.

## Addition and Subtraction:

$a>b \Longrightarrow a \pm k>b \pm k$;
$a \geq b \Longrightarrow a \pm k \geq b \pm k$.
Multiplication and Division:
$a>b \Longrightarrow k a>k b(k>0)$;
$a>b \Longrightarrow k a<k b(k<0)$.

## Squaring:

$a>b$ with $b \geq 0 \Longrightarrow a^{2}>b^{2}$.

## Absolute Values and Inequalities

For any real number $n$, the absolute value of $n$ is defined and denoted by

$$
|n|= \begin{cases}n & \text { if } n>0 \\ -n & \text { if } n<0 \\ 0 & \text { if } n=0\end{cases}
$$

Thus we can write $|x|<n$ as an equivalent way $-n<x<n(n>0)$. Also $|x| \leq n$ if and only if $-n \leq x \leq n(n>0)$.

The following properties characterize absolute values:

1) $|m|+|n| \geq|m+n|$;
2) $|m| \cdot|n|=|m \cdot n|$;
3) $\frac{|m|}{|n|}=\left|\frac{m}{n}\right|$.

## Solution of an Inequality

Example 6.5.1 Find the solution of the inequality $3 x-3>x+1$. By adding $(3-x)$ to both sides, we have

$$
3 x-3+3-x>x+1+3-x .
$$

Thus, $2 x>4$ so $x>2$.

Example 6.5.2 Solve the inequality $|1-x| \leq 3$.
From $|1-x| \leq 3$, we have $-3 \leq 1-x \leq 3$, or $-4 \leq-x \leq 2$. Thus, $4 \geq x \geq-2$, i.e., $-2 \leq x \leq 4$.

### 6.6 Limit Theorems

## Theorems Involving a Single Equation

Theorem I: If $q=a v+b$, then $\lim _{v \rightarrow N} q=a N+b$.
Theorem II: If $q=g(v)=b$, then $\lim _{v \rightarrow N} q=b$.
Theorem III: $\lim _{v \rightarrow N} v^{k}=N^{k}$.

Example 6.6.1 Given $q=5 v+7$, then $\lim _{v \rightarrow 2}=5 \cdot 2+7=17$.
Example 6.6.2 $q=v^{3}$. Find $\lim _{v \rightarrow 2} q$.
By theorem III, we have $\lim _{v \rightarrow 2}=2^{3}=8$.

## Theorems Involving Two Functions

For two functions $q_{1}=g(v)$ and $q_{2}=h(v)$, if $\lim _{v \rightarrow N} q_{1}=L_{1}, \lim _{v \rightarrow N} q_{2}=$ $L_{2}$, then we have the following theorems:

Theorem IV: $\lim _{v \rightarrow N}\left(q_{1}+q_{2}\right)=L_{1}+L_{2}$.
Theorem V: $\lim _{v \rightarrow N}\left(q_{1} q_{2}\right)=L_{1} L_{2}$.
Theorem VI: $\lim _{v \rightarrow N} \frac{q_{1}}{q_{2}}=\frac{L_{1}}{L_{2}}\left(L_{2} \neq 0\right)$.
Example 6.6.3 Find $\lim _{v \rightarrow 0} \frac{1+v}{2+v}$.
Since $\lim _{v \rightarrow 0}(1+v)=1$ and $\lim _{v \rightarrow 0}(2+v)=2$, so $\lim _{v \rightarrow 0} \frac{1+v}{2+v}=\frac{1}{2}$.
Remark 6.6.1 Note that $L_{1}$ and $L_{2}$ represent finite numbers; otherwise theorems do not apply.

## Limit of a Polynomial Function

$$
\lim _{v \rightarrow N} a_{0}+a_{1} v+a_{2} v^{2}+\cdots+a_{n} v^{n}=a_{0}+a_{1} N+a_{2} N^{2}+\cdots+a_{n} N^{n} .
$$

### 6.7 Continuity and Differentiability of a Function

## Continuity of a Function

Definition 6.7.1 A function $q=g(v)$ is said to be continuous at $N$ if $\lim _{v \rightarrow N} q$ exists and $\lim _{v \rightarrow N} g(v)=g(N)$.

Thus the term continuous involves no less than three requirements: (1) the point $N$ must be in the domain of the function; (2) $\lim _{v \rightarrow N} g(v)$ exists; and (3) $\lim _{v \rightarrow N} g(v)=g(N)$.

Remark 6.7.1 It is important to note that while - in discussing the limit of a function - the point $(N, L)$ is excluded from consideration, we are no longer excluding it in defining continuity at point $N$. Rather, as the third requirement specifically states, the point $(N, L)$ must be on the graph of the function before the function can be considered as continuous at point $N$.

## Polynomial and Rational Functions

From the discussion of the limit of polynomial function, we know that the limit exists and equals the value of the function at $N$. Since $N$ is a point in the domain of the function, we can conclude that any polynomial function is continuous in its domain. By those theorems involving two functions, we also know any rational function is continuous in its domain.

Example 6.7.1 $q=\frac{4 v^{2}}{v^{2}+1}$.
Then

$$
\lim _{v \rightarrow N} \frac{4 v^{2}}{v^{2}+1}=\frac{\lim _{v \rightarrow N} 4 v^{2}}{\lim _{v \rightarrow N}\left(v^{2}+1\right)}=\frac{4 N^{2}}{N^{2}+1} .
$$

Example 6.7.2 The rational function

$$
q=\frac{v^{3}+v^{2}-4 v-4}{v^{2}-4}
$$

is not defined at $v=2$ and $v=-2$. Since $v=2,-2$ are not in the domain, the function is discontinuous at $v=2$ and $v=-2$, despite the fact that its limit exists as $v \rightarrow 2$ or -2 by noting

$$
\begin{aligned}
q & =\frac{v^{3}+v^{2}-4 v-4}{v^{2}-4}=\frac{v\left(v^{2}-4\right)+v^{2}-4}{v^{2}-4} \\
& =\frac{(v+1)\left(v^{2}-4\right)}{v^{2}-4}=v+1 \quad(v \neq 2,-2) .
\end{aligned}
$$

## Differentiability Implies Continuity

By the definition of the derivative of a function $y=f(x)$, we know that $f^{\prime}\left(x_{0}\right)$ exists at $x_{0}$ if the lim of $\Delta y / \Delta x$ exists at $x=x_{0}$ as $\Delta x \rightarrow 0$, i.e.,

$$
\begin{aligned}
f^{\prime}\left(x_{0}\right) & =\lim _{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} \\
& \equiv \lim _{\Delta x \rightarrow 0} \frac{f\left(x_{0}+\Delta x\right)-f\left(x_{0}\right)}{\Delta x} \text { (differentiability condition). }
\end{aligned}
$$

On the other hand, the function $y=f(x)$ is continuous at $x_{0}$ if and only if

$$
\lim _{x \rightarrow x_{0}} f(x)=f\left(x_{0}\right) \text { (continuity condition). }
$$

We want to know what is the relationship between the continuity and differentiability of a function. Now we show the continuity of $f$ is a necessary condition for its differentiability. But this is not sufficient.

Since the notation $x \rightarrow x_{0}$ implies $x \neq x_{0}$, so $x-x_{0}$ is a nonzero number, it is permissible to write the following identity:

$$
f(x)-f\left(x_{0}\right)=\frac{f(x)-f\left(x_{0}\right)}{x-x_{0}}\left(x-x_{0}\right)
$$

Taking the limit of each side of the above equation as $x \rightarrow x_{0}$ yields the following results:

$$
\begin{aligned}
& \text { Left side }=\lim _{x \rightarrow x_{0}}\left(f(x)-f\left(x_{0}\right)\right)=\lim _{x \rightarrow x_{0}} f(x)-f\left(x_{0}\right) . \\
& \qquad \begin{aligned}
\text { Right side } & =\lim _{x \rightarrow x_{0}} \frac{f(x)-f\left(x_{0}\right)}{x-x_{0}} \lim _{x \rightarrow x_{0}}\left(x-x_{0}\right) \\
& =f^{\prime}\left(x_{0}\right) \lim _{x \rightarrow x_{0}}\left(x-x_{0}\right)=0 .
\end{aligned}
\end{aligned}
$$

Thus $\lim _{x \rightarrow x_{0}} f(x)-f\left(x_{0}\right)=0$. So $\lim _{x \rightarrow x_{0}} f(x)=f\left(x_{0}\right)$ which means $f(x)$ is continuous at $x=x_{0}$.

Although differentiability implies continuity, the converse may not be true. That is, continuity is a necessary, but not sufficient, condition for differentiability. The following example shows this.

Example 6.7.3 $f(x)=|x|$.
This function is clearly continuous at $x=0$. Now we show that it is not differentiable at $x=0$. This involves the demonstration of a disparity between the left-side limit and the right-side limit. Since, in considering the right-side limit $x>0$, we have

$$
\lim _{x \rightarrow 0^{+}} \frac{f(x)-f(0)}{x-0}=\lim _{x \rightarrow 0^{+}} \frac{x}{x}=\lim _{x \rightarrow 0^{+}} 1=1 .
$$

On the other hand, in considering the left-side limit, $x<0$; we have

$$
\lim _{x \rightarrow 0^{-}} \frac{f(x)-f(0)}{x-0}=\lim _{x \rightarrow 0^{-}} \frac{|x|}{x}=\lim _{x \rightarrow 0^{-}} \frac{-x}{x}=\lim _{x \rightarrow 0^{-}}-1=-1
$$

Thus, $\lim _{x \rightarrow 0} \frac{\Delta y}{\Delta x}$ does not exist since the left-side limit and the rightside limit are not the same, which implies that the derivative of $y=|x|$ does not exist at $x=0$.

## Chapter 7

## Rules of Differentiation and Their Use in Comparative Statics

The central problem of comparative-static analysis is to find the rate of change, which can be identified with finding the derivative of a function $y=f(x)$ when only a small change in $x$ is being considered. To begin our study of comparative-static models, let us first review some rules of differentiation.

### 7.1 Rules of Differentiation for a Function of One Variable

## Constant-Function Rule

If $y=f(x)=c$, where $c$ is a constant, then

$$
\frac{d y}{d x} \equiv y^{\prime} \equiv f^{\prime}=0
$$

Proof.

$$
\frac{d y}{d x}=\lim _{x^{\prime} \rightarrow x} \frac{f\left(x^{\prime}\right)-f(x)}{x^{\prime}-x}=\lim _{x^{\prime} \rightarrow x} \frac{c-c}{x^{\prime}-x}={ }_{x^{\prime} \rightarrow x} 0=0 .
$$

We can also write $\frac{d y}{d x}=\frac{d f}{d x}$ as

$$
\frac{d}{d x} y=\frac{d}{d x} f
$$

So we may consider $d / d x$ as an operator symbol.

## Power-Function Rule

If $y=f(x)=x^{a}$ where $a$ is any real number $-\infty<a<\infty$,

$$
\frac{d}{d x} f(x)=a x^{a-1}
$$

Remark 7.1.1 Note that:
(i) If $a=0$, then

$$
\frac{d}{d x} x^{0}=\frac{d}{d x} 1=0
$$

(ii) If $a=1$, then $y=x$. Thus

$$
\frac{d x}{d x}=1
$$

For simplicity, we prove this rule only for the case where $a=n$, where $n$ is any positive integer. It can be verified that

$$
x^{n}-x_{0}^{n}=\left(x-x_{0}\right)\left(x^{n-1}+x_{0} x^{n-2}+x_{0}^{2} x^{n-3}+\cdots+x_{0}^{n-1}\right) .
$$

Then

$$
\frac{x^{n}-x_{0}^{n}}{x-x_{0}}=x^{n-1}+x_{0} x^{n-2}+x_{0}^{2} x^{n-3}+\cdots+x_{0}^{n-1}
$$

Therefore,

$$
\begin{aligned}
f^{\prime}\left(x_{0}\right) & =\lim _{x \rightarrow x_{0}} \frac{f(x)-f\left(x_{0}\right)}{x-x_{0}}=\lim _{x \rightarrow x_{0}} \frac{x^{n}-x_{0}^{n}}{x-x_{0}} \\
& =\lim _{x \rightarrow x_{0}} x^{n-1}+x_{0} x^{n-2}+x_{0}^{2} x^{n-3}+\cdots+x_{0}^{n-1} \\
& =x_{0}^{n-1}+x_{0}^{n-1}+x_{0}^{n-1}+\cdots+x_{0}^{n-1} \\
& =n x_{0}^{n-1} .
\end{aligned}
$$

Example 7.1.1 Suppose $y=f(x)=x^{-3}$. Then $y^{\prime}=-3 x^{-4}$.

Example 7.1.2 Suppose $y=f(x)=\sqrt{x}$. Then $y^{\prime}=\frac{1}{2} x^{-\frac{1}{2}}$. In particular, we can know that $f^{\prime}(2)=\frac{1}{2} \cdot 2^{-\frac{1}{2}}=\frac{\sqrt{2}}{4}$.

## Power-Function Rule Generalized

If the function is given by $y=c x^{a}$, then

$$
\frac{d y}{d x}=\frac{d f}{d x}=a c x^{a-1}
$$

Example 7.1.3 Suppose $y=2 x$. Then

$$
\frac{d y}{d x}=2 x^{0}=2
$$

Example 7.1.4 Suppose $y=4 x^{3}$. Then

$$
\frac{d y}{d x}=4 \cdot 3 x^{3-1}=12 x^{2} .
$$

Example 7.1.5 Suppose the function $y=3 x^{-2}$. Then

$$
\frac{d y}{d x}=-6 x^{-2-1}=-6 x^{-3}
$$

## Common Rules:

$$
\begin{aligned}
& f(x)=\text { constant } \Rightarrow f^{\prime}(x)=0 ; \\
& f(x)=x^{a}(a \text { is constant }) \Rightarrow f^{\prime}(x)=a x^{a-1} ; \\
& f(x)=e^{x} \Rightarrow f^{\prime}(x)=e^{x} ; \\
& f(x)=a^{x}(a>0) \Rightarrow f^{\prime}(x)=a^{x} \ln a ; \\
& f(x)=\ln x \Rightarrow f^{\prime}(x)=\frac{1}{x} \\
& f(x)=\log _{a} x(a>0 ; a \neq 1) \Rightarrow f^{\prime}(x)=\frac{1}{x} \log _{a} e=\frac{1}{x \ln a} ; \\
& f(x)=\sin x \Rightarrow f^{\prime}(x)=\cos x ; \\
& f(x)=\cos x \Rightarrow f^{\prime}(x)=-\sin x ; \\
& f(x)=\tan x \Rightarrow f^{\prime}(x)=\frac{1}{\cos ^{2} x} ; \\
& f(x)=\operatorname{ctan} x \Rightarrow f^{\prime}(x)=-\frac{1}{\sin ^{2} x} ; \\
& f(x)=\arcsin x \Rightarrow f^{\prime}(x)=\frac{1}{\sqrt{1-x^{2}}} ; \\
& f(x)=\arccos x \Rightarrow f^{\prime}(x)=-\frac{1}{\sqrt{1-x^{2}}} ; \\
& f(x)=\arctan x \Rightarrow f^{\prime}(x)=\frac{1}{1-x^{2}} ; \\
& f(x)=\operatorname{arcctan} x \Rightarrow f^{\prime}(x)=-\frac{1}{1-x^{2}} ;
\end{aligned}
$$

### 7.2 Rules of Differentiation Involving Two or More Functions of the Same Variable

Let $f(x)$ and $g(x)$ be two differentiable functions. We have the following rules:

### 7.2. RULES OF DIFFERENTIATION INVOLVING TWO OR MORE FUNCTIONS OF THE SAME

## Sum-Difference Rule:

$$
\frac{d}{d x}[f(x) \pm g(x)]=\frac{d}{d x} f(x) \pm \frac{d}{d x} g(x)=f^{\prime}(x) \pm g^{\prime}(x)
$$

This rule can easily be extended to more functions:

$$
\frac{d}{d x}\left[\sum_{i=1}^{n} f_{i}(x)\right]=\sum_{i=1}^{n} \frac{d}{d x} f_{i}(x)=\sum_{i=1}^{n} f_{i}^{\prime}(x)
$$

Example 7.2.1 Consider the function $a x^{2}+b x+c$. We have

$$
\frac{d}{d x}\left(a x^{2}+b x+c\right)=2 a x+b .
$$

Example 7.2.2 Suppose that a short-run total-cost function is given by $c=$ $Q^{3}-4 Q^{2}+10 Q+75$. Then the marginal-cost function is the limit of the quotient $\Delta C / \Delta Q$, or the derivative of the $C$ function:

$$
\frac{d C}{d Q}=3 Q^{2}-8 Q+10
$$

In general, if a primitive function $y=f(x)$ represents a total function, then the derivative function $d y / d x$ represents its marginal function. The derivative of a function is the slope of its curve, so the marginal function shows the slope of the curve of the total function at each point $x$.

## L'Hopital's Rule

We can use derivatives to find the limit of a continuous function when both the numerator and denominator approach zero (or both approach infinity). This is known as the L'Hopital's rule, which states:

Theorem 7.2.1 (L'Hopital's Rule) Suppose that $f(x)$ and $g(x)$ are differentiable on an open interval $(a, b)$, except possibly at $c$. If $\lim _{x \rightarrow c} f(x)=\lim _{x \rightarrow c} g(x)=$

0 or $\lim _{x \rightarrow c} f(x)=\lim _{x \rightarrow c} g(x)= \pm \infty, g^{\prime}(x) \neq 0$ for all $x$ in $(a, b)$ with $x \neq c$, and $\lim _{x \rightarrow c} \frac{f^{\prime}(x)}{g^{\prime}(x)}$ exists, then

$$
\lim _{x \rightarrow c} \frac{f(x)}{g(x)}=\lim _{x \rightarrow c} \frac{f^{\prime}(x)}{g^{\prime}(x)} .
$$

Example 7.2.3 Consider the function

$$
q=\frac{v^{3}+v^{2}-4 v-4}{v^{2}-4}
$$

We have $\lim _{v \rightarrow 2}\left(v^{3}+v^{2}-4 v-4\right)=0$ and $\lim _{v \rightarrow 2}\left(v^{2}-4\right)=0$. By applying L'Hopital's Rule, we get

$$
\begin{aligned}
\lim _{v \rightarrow 2} \frac{v^{3}+v^{2}-4 v-4}{v^{2}-4} & =\lim _{v \rightarrow 2} \frac{\frac{d}{d v}\left(v^{3}+v^{2}-4 v-4\right)}{\frac{d}{d v}\left(v^{2}-4\right)} \\
& =\lim _{v \rightarrow 2} \frac{3 v^{2}+2 v-4}{2 v}=3
\end{aligned}
$$

Example 7.2.4 Consider the function

$$
q=\frac{4 v+5}{v^{2}+2 v-3}
$$

As $\lim _{v \rightarrow \infty} 4 v+5=\infty$ and $\lim _{v \rightarrow \infty} v^{2}+2 v-3=\infty$, by L'Hopital Rule,

$$
\begin{aligned}
\lim _{v \rightarrow \infty} \frac{4 v+5}{v^{2}+2 v-3} & =\lim _{v \rightarrow \infty} \frac{\frac{d}{d v}(4 v+5)}{\frac{d}{d v}\left(v^{2}+2 v-3\right)} \\
& =\lim _{v \rightarrow \infty} \frac{4}{2 v+2}=0
\end{aligned}
$$

## Product Rule:

$$
\begin{aligned}
\frac{d}{d x}[f(x) g(x)] & =f(x) \frac{d}{d x} g(x)+g(x) \frac{d}{d x} f(x) \\
& =f(x) g^{\prime}(x)+g(x) f^{\prime}(x)
\end{aligned}
$$

Proof.

$$
\begin{aligned}
\frac{d}{d x}\left[f\left(x_{0}\right) g\left(x_{0}\right)\right] & =\lim _{x \rightarrow x_{0}} \frac{f(x) g(x)-f\left(x_{0}\right) g\left(x_{0}\right)}{x-x_{0}} \\
& =\lim _{x \rightarrow x_{0}} \frac{f(x) g(x)-f(x) g\left(x_{0}\right)+f(x) g\left(x_{0}\right)-f\left(x_{0}\right) g\left(x_{0}\right)}{x-x_{0}} \\
& =\lim _{x \rightarrow x_{0}} \frac{f(x)\left[g(x)-g\left(x_{0}\right)\right]+g\left(x_{0}\right)\left[f(x)-f\left(x_{0}\right)\right]}{x-x_{0}} \\
& =\lim _{x \rightarrow x_{0}} f(x) \frac{g(x)-g\left(x_{0}\right)}{x-x_{0}}+\lim _{x \rightarrow x_{0}} g\left(x_{0}\right) \frac{f(x)-f\left(x_{0}\right)}{x-x_{0}} \\
& =f\left(x_{0}\right) g^{\prime}\left(x_{0}\right)+g\left(x_{0}\right) f^{\prime}\left(x_{0}\right) .
\end{aligned}
$$

Since this is true for any $x=x_{0}$, this proves the rule.
Example 7.2.5 Suppose $y=(2 x+3)\left(3 x^{2}\right)$. Let $f(x)=2 x+3$ and $g(x)=3 x^{2}$.
Then $f^{\prime}(x)=2, g^{\prime}(x)=6 x$. Hence,

$$
\begin{aligned}
\frac{d}{d x}\left[(2 x+3)\left(3 x^{2}\right)\right] & =(2 x+3) 6 x+3 x^{2} \cdot 2 \\
& =12 x^{2}+18 x+6 x^{2} \\
& =18 x^{2}+18 x .
\end{aligned}
$$

As an extension of the rule to the case of three functions, we have

$$
\frac{d}{d x}[f(x) g(x) h(x)]=f^{\prime}(x) g(x) h(x)+f(x) g^{\prime}(x) h(x)+f(x) g(x) h^{\prime}(x)
$$

## Finding Marginal-Revenue Function from Average-Revenue Function

Suppose that the average-revenue (AR) function is specified by

$$
A R=15-Q
$$

The total-revenue (TR) function is

$$
T R \equiv A R \cdot Q=15 Q-Q^{2}
$$

Then, the marginal-revenue (MR) function is given by

$$
M R \equiv \frac{d}{d Q} T R=15-2 Q
$$

In general, if $A R=f(Q)$, then

$$
T R \equiv A R \cdot Q=Q f(Q)
$$

Thus

$$
M R \equiv \frac{d}{d Q} T R=f(Q)+Q f^{\prime}(Q)
$$

From this, we can tell the relationship between $M R$ and $A R$. Since

$$
M R-A R=Q f^{\prime}(Q)
$$

they will always differ the amount of $Q f^{\prime}(Q)$. Also, since

$$
A R \equiv \frac{T R}{Q}=\frac{P Q}{Q}=P
$$

we can view $A R$ as the inverse demand function for the product of the firm. If the market is perfectly competitive, i.e., the firm takes the price as given, then $P=f(Q)=$ constant. Hence $f^{\prime}(Q)=0$. Thus $M R-A R=0$ or $M R=A R$. Under imperfect competition, on the other hand, the $A R$ curve is normally downward-sloping, so that $f^{\prime}(Q)<0$. Thus $M R<A R$.

## Quotient Rule

$$
\frac{d}{d x}\left[\frac{f(x)}{g(x)}\right]=\frac{f^{\prime}(x) g(x)-f(x) g^{\prime}(x)}{g^{2}(x)}
$$

We will come back to prove this rule after learning the chain rule.

## Example 7.2.6

$$
\begin{gathered}
\frac{d}{d x}\left[\frac{2 x-3}{x+1}\right]=\frac{2(x+1)-(2 x-3)(1)}{(x+1)^{2}}=\frac{5}{(x+1)^{2}} . \\
\frac{d}{d x}\left[\frac{5 x}{x^{2}+1}\right]=\frac{5\left(x^{2}+1\right)-5 x(2 x)}{\left(x^{2}+1\right)^{2}}=\frac{5\left(1-x^{2}\right)}{\left(x^{2}+1\right)^{2}} . \\
\frac{d}{d x}\left[\frac{a x^{2}+b}{c x}\right]=\frac{2 a x(c x)-\left(a x^{2}+b\right) c}{(c x)^{2}}=\frac{c\left(a x^{2}-b\right)}{(c x)^{2}}=\frac{a x^{2}-b}{c x^{2}} .
\end{gathered}
$$

## Relationship Between Marginal-Cost and Average-Cost Functions

As an economic application of the quotient rule, let us consider the rate of change of average cost when output varies.

Given a total cost function $C=C(Q)$, the average cost $(A C)$ function and the marginal-cost ( $M C$ ) function are given by

$$
A C \equiv \frac{C(Q)}{Q}(Q>0)
$$

and

$$
M C \equiv C^{\prime}(Q)
$$

The rate of change of $A C$ with respect to $Q$ can be found by differentiating $A C$ :

$$
\begin{aligned}
\frac{d}{d Q} A C(Q) & =\frac{d}{d Q}\left[\frac{C(Q)}{Q}\right] \\
& =\frac{C^{\prime}(Q) Q-C(Q)}{Q^{2}} \\
& =\frac{1}{Q}\left[C^{\prime}(Q)-\frac{C(Q)}{Q}\right] \\
& =\frac{1}{Q}[M C(Q)-A C(Q)] .
\end{aligned}
$$



Figure 7.1: Graphical representation of relationship between marginal-cost and average-cost functions

From this it follows that, for $Q>0$, we have:

$$
\begin{aligned}
& \frac{d}{d Q} A C>0 \text { iff } M C(Q)>A C(Q) ; \\
& \frac{d}{d Q} A C=0 \text { iff } M C(Q)=A C(Q) \\
& \frac{d}{d Q} A C<0 \text { iff } M C(Q)<A C(Q) .
\end{aligned}
$$

### 7.3 Rules of Differentiation Involving Functions of Different Variables

Now we consider cases where there are two or more differentiable functions, each of which has a distinct independent variable.

### 7.3. RULES OF DIFFERENTIATION INVOLVING FUNCTIONS OF DIFFERENT VARIABLES111

## Chain Rule

If we have a function $z=f(y)$, where $y$ is in turn a function of another variable $x$, say, $y=g(x)$, then the derivative of $z$ with respect to $x$ is given by Chain Rule:

$$
\frac{d z}{d x}=\frac{d z}{d y} \cdot \frac{d y}{d x}=f^{\prime}(y) g^{\prime}(x) .
$$

The chain rule appeals easily to intuition. Given a $\Delta x$, there must result in a corresponding $\Delta y$ via the function $y=g(x)$, but this $\Delta y$ will in turn being about a $\Delta z$ via the function $z=f(y)$.

Proof. Note that

$$
\frac{d z}{d x}=\lim _{\Delta x \rightarrow 0} \frac{\Delta z}{\Delta x}=\lim _{\Delta x \rightarrow 0} \frac{\Delta z}{\Delta y} \frac{\Delta y}{\Delta x} .
$$

Since $\Delta x \rightarrow 0$ implies $\Delta y \rightarrow 0$ which in turn implies $\Delta z \rightarrow 0$, we then have

$$
\frac{d z}{d x}=\frac{d z}{d y} \cdot \frac{d y}{d x}=f^{\prime}(y) g^{\prime}(x) . \text { Q.E.D. }
$$

In view of the function $y=g(x)$, we can express the function $z=f(y)$ as $z=f(g(x))$, where the contiguous appearance of the two function symbols $f$ and $g$ indicates that this is a compose function (function of a function). So sometimes, the chain rule is also known as the composite function rule.

As an application of this rule, we use it to prove the quotient rule.
For $z=\frac{1}{g(x)}$, let $y=g(x)$. Then $z=\frac{1}{y}=y^{-1}$ and thus by the chain rule, we have

$$
\frac{d z}{d x}=\frac{d z}{d y} \cdot \frac{d y}{d x}=-\frac{1}{y^{2}} g^{\prime}(x)=-\frac{g^{\prime}(x)}{g^{2}(x)} .
$$

Therefore, we have:

$$
\begin{aligned}
\frac{d}{d x}\left[\frac{f(x)}{g(x)}\right] & =\frac{d}{d x}\left[f(x) \cdot g^{-1}(x)\right] \\
& =f^{\prime}(x) g^{-1}(x)+f(x) \frac{d}{d x}\left[g^{-1}(x)\right] \\
& =f^{\prime}(x) g^{-1}(x)+f(x)\left[-\frac{g(x)}{g^{2}(x)}\right] \\
& =\frac{f^{\prime}(x) g(x)-f(x) g^{\prime}(x)}{g^{2}(x)} \cdot Q . E . D .
\end{aligned}
$$

Example 7.3.1 If $z=3 y^{2}$ and $y=2 x+5$, then

$$
\frac{d z}{d x}=\frac{d z}{d y} \frac{d y}{d x}=6 y(2)=12 y=12(2 x+5)
$$

Example 7.3.2 If $z=y-3$ and $y=x^{3}$, then

$$
\frac{d z}{d x}=\frac{d z}{d y} \frac{d y}{d x}=1 \cdot 3 x^{2}=3 x^{2}
$$

The usefulness of this rule can best be appreciated when one must differentiate a function such as those below.

Example 7.3.3 $z=\left(x^{2}+3 x-2\right)^{17}$. Let $z=y^{17}$ and $y=x^{2}+3 x-2$.

$$
\begin{aligned}
\frac{d z}{d x} & =\frac{d z}{d y} \frac{d y}{d x}=17 y^{16} \cdot(2 x+3) \\
& =17\left(x^{2}+3 x-2\right)^{16}(2 x+3)
\end{aligned}
$$

Once being familiar with the chain rule, it is unnecessary to adopt intermediate variables to find the derivative of a function.

We can find the derivative of a more general function by applying the chain rule repeatedly.

### 7.3. RULES OF DIFFERENTIATION INVOLVING FUNCTIONS OF DIFFERENT VARIABLES113

Example 7.3.4 $z=\left[\left(x^{3}-2 x+1\right)^{3}+3 x\right]^{-2}$. Applying the chain rule repeatedly, we have.

$$
\frac{d z}{d x}=-2\left[\left(x^{3}-2 x+1\right)^{3}+3 x\right]^{-3}\left[3\left(x^{3}-2 x+1\right)^{2}\left(3 x^{2}-2\right)+3\right] .
$$

Example 7.3.5 Suppose $T R=f(Q)$, where output $Q$ is a function of labor input $L$, or $Q=g(L)$. Then, by the chain rule, the marginal revenue product of labor $\left(M R P_{L}\right)$ is

$$
M R P_{L}=\frac{d R}{d L}=\frac{d R}{d Q} \frac{d Q}{d L}=f^{\prime}(Q) g^{\prime}(L)=M R \cdot M P_{L},
$$

where $M R P_{L}$ is marginal physical product of labor. Thus the result shown above constitutes the mathematical statement of the well-known result in economics that $M R P_{L}=M R \cdot M P_{L}$.

## Inverse-Function Rule

Assume $y=f(x)$ is a one-to-one mapping, i.e., the function with a different value of $x$ will always yield a different value of $y$. Then the function $f$ will have an inverse function $x=f^{-1}(y)$, where the symbol $f^{-1}$ is a function symbol which signifies a function related to the function $f$; it does not mean the reciprocal of the function $f(x)$. When $x$ and $y$ refer specifically to numbers, the property of one-to-one mapping is unique to the class of functions known as monotonic functions.

Definition 7.3.1 A function $f$ is said to be monotonically increasing (decreasing) if $x_{1}>x_{2}$ implies $f\left(x_{1}\right)>f\left(x_{2}\right)$ (resp. $f\left(x_{1}\right)<f\left(x_{2}\right)$ ).

In either of these cases, an inverse function $f^{-1}$ exists.
A practical way of ascertaining the monotonicity of a given function $y=f(x)$ is to check whether $f^{\prime}(x)$ always adheres to the same algebraic
sign for all values of $x$. Geometrically, this means that the slope of the function is either always upward or always downward.

Example 7.3.6 Suppose $y=5 x+25$. Since $y^{\prime}=5$ for all $x$, the function is monotonic and thus the inverse function exists. In fact, it is given by $x=\frac{1}{5}(y-25)$.

If an inverse function exists, the original and the inverse functions must both be monotonic. Moreover, if $f^{-1}$ is the inverse function of $f$, then $f$ must be the inverse function of $f^{-1}$.

In general, we may not have an explicit expression for the inverse function. However, we can easily find the derivative of an inverse function by using the following inverse function rule:

$$
\frac{d x}{d y}=\frac{1}{\frac{d y}{d x}}
$$

Proof:

$$
\frac{d x}{d y}=\lim _{\Delta y \rightarrow 0} \frac{\Delta x}{\Delta y}=\lim _{\Delta x \rightarrow 0} \frac{1}{\frac{\Delta y}{\Delta x}}=\frac{1}{y^{\prime}}
$$

by noting that $\Delta y \rightarrow 0$ implies $\Delta x \rightarrow 0$.

Example 7.3.7 Suppose $y=x^{5}+x$. Then

$$
\frac{d x}{d y}=\frac{1}{\frac{d y}{d x}}=\frac{1}{5 x^{4}+1} .
$$

Example 7.3.8 Given $y=\ln x$, its inverse is $x=e^{y}$. Therefore, by the inverse-function rule, we have

$$
\frac{d x}{d y}=\frac{1}{d y / d x}=\frac{1}{1 / x}=x=e^{y}
$$

### 7.4 Integration (The Case of One Variable)

Let $f(x)$ be a continuous function. The indefinite integral of f (denoted by $\left.\int f(x) d x\right)$ is defined as

$$
\int f(x) d x=F(x)+C
$$

where $F(x)$ is such that $F^{\prime}(x)=f(x)$, and $C$ is an arbitrary constant.

## Rules of Integration

- $\int[a f(x)+b g(x)] d x=a \int f(x) d x+b \int g(x) d x$, where $a$ and $b$ are constants (linearity of the integral);
- $\int f^{\prime}(x) g(x) d x=f(x) g(x)-\int f(x) g^{\prime}(x) d x$ (integration by parts);
- $\int f(u(t)) \frac{d u}{d t} d t=\int f(u) d u$ (integration by substitution).


## Some Special Rules of Integration:

$$
\begin{aligned}
& \int \frac{f^{\prime}(x)}{f(x)} d x=\ln |f(x)|+C ; \\
& \int \frac{1}{x} d x=\ln |x|+C ; \\
& \int e^{x} d x=e^{x}+C ; \\
& \int f^{\prime}(x) e^{f(x)} d x=e^{f(x)}+C ; \\
& \int x^{a} d x=\frac{x^{a+1}}{a+1}+C, a \neq-1 ; \\
& \int a^{x} d x=\frac{a^{x}}{\ln a}+C, a>0 .
\end{aligned}
$$

## Example 7.4.1

$$
\int \frac{x^{2}+2 x+1}{x} d x=\int x d x+\int 2 d x+\int \frac{1}{x} d x=\frac{x^{2}}{2}+2 x+\ln |x|+C .
$$

## Example 7.4.2

$$
\int x e^{-x^{2}} d x=-\frac{1}{2} \int(-2 x) e^{-x^{2}} d x=-\frac{1}{2} \int e^{-z} d z=-\frac{e^{-x^{2}}}{2}+C
$$

## Example 7.4.3

$$
\int x e^{x} d x=x e^{x}-\int e^{x} d x=x e^{x}-e^{x}+C
$$

Definition 7.4.1 (The Newton-Leibniz formula) The definite integral of a continuous function $f$ is

$$
\int_{a}^{b} f(x) d x=\left.F(x)\right|_{a} ^{b}=F(b)-F(a)
$$

for $F(x)$ such that $F^{\prime}(x)=f(x)$ for all $x \in[a, b]$.

Remark 7.4.1 The indefinite integral is a function. The definite integral is a number.

## Properties of Definite Integrals:

$$
\begin{aligned}
& \int_{a}^{b}[\alpha f(x)+\beta g(x)] d x=\alpha \int_{a}^{b} f(x) d x+\beta \int_{a}^{b} g(x) d x \\
& \int_{a}^{b} f(x) d x=-\int_{b}^{a} f(x) d x \\
& \int_{a}^{a} f(x) d x=0 \\
& \int_{a}^{b} f(x) d x=\int_{a}^{c} f(x) d x+\int_{c}^{b} f(x) d x \\
& \left|\int_{a}^{b} f(x) d x\right| \leq \int_{a}^{b}|f(x)| d x ; \\
& \int_{a}^{b} f(x) g^{\prime}(x) d x=\int_{g(a)}^{g(b)} f(u) d u, u=g(x)(\text { change of variable); } \\
& \int_{a}^{b} f^{\prime}(x) g(x) d x=\left.f(x) g(x)\right|_{a} ^{b}-\int_{a}^{b} f(x) g^{\prime}(x) d x
\end{aligned}
$$

where $a, b, c, \alpha, \beta$ are real numbers.

## Some More Useful Results:

$$
\frac{d}{d \lambda} \int_{a(\lambda)}^{b(\lambda)} f(x) d x=f(b(\lambda)) b^{\prime}(\lambda)-f(a(\lambda)) a^{\prime}(\lambda)
$$

## Example 7.4.4

$$
\frac{d}{d x} \int_{a}^{x} f(t) d t=f(x)
$$

### 7.5 Partial Differentiation

So far, we have considered only the derivative of functions of a single independent variable. However, in many economic models, several parameters appear, and the equilibrium value of each endogenous variable may
be a function of more than one parameter. Therefore, we now consider the derivative of a function of more than one variable.

## Partial Derivatives

Consider a function

$$
y=f\left(x_{1}, x_{2}, \cdots, x_{n}\right)
$$

where the variables $x_{i}(i=1,2, \cdots, n)$ are all independent of each other, so that each can vary by itself without affecting the others. If the variable $x_{i}$ changes by $\Delta x_{i}$ while the other variables remain fixed, there will be a corresponding change in $y$, namely, $\Delta y$. The difference quotient in this case can be expressed as

$$
\frac{\Delta y}{\Delta x_{i}}=\frac{f\left(x_{1}, x_{2}, \cdots, x_{i-1}, x_{i}+\Delta x_{i}, x_{i}, \cdots, x_{n}\right)-f\left(x_{1}, x_{2}, \cdots, x_{n}\right)}{\Delta x_{i}} .
$$

If we take the limit of $\Delta y / \Delta x_{i}$, that limit will constitute a derivative. We call it the partial derivative of $y$ with respect to $x_{i}$. The process of taking partial derivatives is called partial differentiation. Denote the partial derivative of $y$ with respect to $x_{i}$ by $\frac{\partial y}{\partial x_{i}}$, i.e.,

$$
\frac{\partial y}{\partial x_{i}}=\lim _{\Delta x_{i} \rightarrow 0} \frac{\Delta y}{\Delta x_{i}} .
$$

We can use $f_{i}$ to denote $\partial y / \partial x_{i}$. If the function happens to be written in terms of unsubscripted variables, such as $y=f(u, v, w)$, one also uses $f_{u}, f_{v}$, and $f_{w}$ to denote the partial derivatives.

Partial differentiation differs from the previously discussed differentiation primarily in that we must hold the other independent variables constant while allowing one variable to vary.

Example 7.5.1 Suppose that $y=f\left(x_{1}, x_{2}\right)=3 x_{1}^{2}+x_{1} x_{2}+4 x_{2}^{2}$. Find $\partial y / \partial x_{1}$ and $\partial y / \partial x_{2}$.

$$
\begin{aligned}
\frac{\partial y}{\partial x_{1}} & \equiv \frac{\partial f}{\partial x_{1}}=6 x_{1}+x_{2} \\
\frac{\partial y}{\partial x_{2}} & \equiv \frac{\partial f}{\partial x_{2}}=x_{1}+8 x_{2}
\end{aligned}
$$

Example 7.5.2 For $y=f(u, v)=(u+4)(3 u+2 v)$, we have

$$
\begin{aligned}
\frac{\partial y}{\partial u} \equiv f_{u} & =(3 u+2 v)+(u+4) \cdot 3 \\
& =6 u+2 v+12 \\
\frac{\partial y}{\partial v} & \equiv f_{v}=2(u+4)
\end{aligned}
$$

When $u=2$ and $v=1$, then $f_{u}(2,1)=26$ and $f_{v}(2,1)=12$.
Example 7.5.3 Given $y=(3 u-2 v) /\left(u^{2}+3 v\right)$,

$$
\frac{\partial y}{\partial u}=\frac{3\left(u^{2}+3 v\right)-(3 u-2 v)(2 u)}{\left(u^{2}+3 v\right)^{2}}=\frac{-3 u^{2}+4 u v+9 v}{\left(u^{2}+3 v\right)^{2}}
$$

and

$$
\frac{\partial y}{\partial v}=\frac{-2\left(u^{2}+3 v\right)-(3 u-2 v) \cdot 3}{\left(u^{2}+3 v\right)^{2}}=\frac{-u(2 u+9)}{\left(u^{2}+3 v\right)^{2}}
$$

Example 7.5.4 Given utility function $u=x^{a} y^{b}(a>0$ and $b>0)$, find $M U_{x}$ (the marginal utility of $x$ ), $M U_{y}$ (the marginal utility of $y$ ), and $M R S_{x y}$ (the marginal rate of substitution of $x$ for $y$ ):

$$
\begin{aligned}
& M U_{x}=\frac{\partial u}{\partial x}=a x^{a-1} y^{b} \\
& M U_{y}=\frac{\partial u}{\partial y}=b x^{a} y^{b-1}
\end{aligned}
$$

and

$$
M R S_{x y}=\frac{M U_{x}}{M U_{y}}=\frac{a x^{a-1} y^{b}}{b x^{a} y^{b-1}}=\frac{a y}{b x} .
$$

### 7.6 Applications to Comparative-Static Analysis

By understanding the different rules of differentiation, we can now address the issue posed in comparative-static analysis, which is how changes in exogenous variables or parameters affect the equilibrium value of an endogenous variable.

## Market Model

For the one-commodity market model:

$$
\begin{array}{cc}
Q_{d}=a-b p & (a, b>0) \\
Q_{s}=-c+d p & (c, d>0)
\end{array}
$$

the equilibrium price and quantity are given by

$$
\begin{aligned}
& \bar{p}=\frac{a+c}{b+d} \\
& \bar{Q}=\frac{a d-b c}{b+d} .
\end{aligned}
$$

We can refer to the solutions obtained as the reduced form since the expressions for the two endogenous variables are now explicit functions of the four independent variables, $a, b, c$, and $d$.

To determine the impact of a small change in one of the parameters on the value of $\bar{p}$ or $\bar{Q}$, we only need to find the partial derivatives. If we can determine the sign of a partial derivative, we can identify the direction in which $\bar{p}$ will move when a parameter changes. This represents a qualitative conclusion. If we can determine the magnitude of the partial derivative, we can draw a quantitative conclusion.

It is important to distinguish between the two derivatives, such as $\partial \bar{Q} / \partial a$ and $\partial Q_{d} / \partial a$. The latter derivative is appropriate for the demand
function alone, without considering the supply function. The derivative $\partial \bar{Q} / \partial a$, however, reflects the equilibrium quantity that considers the interaction of both demand and supply. To emphasize this distinction, we refer to the partial derivatives of $\bar{p}$ and $\bar{Q}$ with respect to the parameters as comparative-static derivatives.


Figure 7.2: The graphical illustration of comparative statics: (a) increase in $a$; (b) increase in $b$; (c) increase in $c$, and (d) increase in $d$.

For instance, for $\bar{p}$, we have

$$
\begin{aligned}
& \frac{\partial \bar{p}}{\partial a}=\frac{1}{b+d}>0 ; \\
& \frac{\partial \bar{p}}{\partial b}=-\frac{a+c}{(b+d)^{2}}<0 ; \\
& \frac{\partial \bar{p}}{\partial c}=\frac{1}{b+d}>0 ; \\
& \frac{\partial \bar{p}}{\partial d}=-\frac{a+c}{(b+d)^{2}}<0 .
\end{aligned}
$$

## National-Income Model

$$
\begin{aligned}
& Y=C+I_{0}+G_{0} \quad \text { (equilibrium condition); } \\
& C=\alpha+\beta(Y-T) \quad(\alpha>0 ; 0<\beta<1) \\
& T=\gamma+\delta Y \quad(\gamma>0 ; 0<\delta<1)
\end{aligned}
$$

where the endogenous variables are the national income $Y$, consumption $C$, and taxes $T$. The equilibrium income (in reduced form) is

$$
\bar{Y}=\frac{\alpha-\beta \gamma+I_{0}+G_{0}}{1-\beta+\beta \delta}
$$

Thus,

$$
\begin{aligned}
\frac{\partial \bar{Y}}{\partial G_{0}} & =\frac{1}{1-\beta+\beta \delta}>0 \text { (the government-expenditure multiplier) } \\
\frac{\partial \bar{Y}}{\partial \gamma} & =\frac{-\beta}{1-\beta+\beta \delta}<0 \\
\frac{\partial \bar{Y}}{\partial \delta} & =\frac{-\beta\left(\alpha-\beta \gamma+I_{0}+G_{0}\right)}{(1-\beta+\beta \delta)^{2}}=\frac{-\beta \bar{Y}}{(1-\beta+\beta \delta)}<0
\end{aligned}
$$

### 7.7 Note on Jacobian Determinants

Partial derivatives can also provide a means of testing whether there exists functional (linear or nonlinear) dependence among a set of $n$ variables. This is related to the notion of Jacobian determinants.

Consider $n$ differentiable functions in $n$ variables not necessary linear.

$$
\begin{gathered}
y_{1}=f^{1}\left(x_{1}, x_{2}, \cdots, x_{n}\right) ; \\
y_{2}=f^{2}\left(x_{1}, x_{2}, \cdots, x_{n}\right) ; \\
\cdots ; \\
y_{n}=f^{n}\left(x_{1}, x_{2}, \cdots, x_{n}\right),
\end{gathered}
$$

where the symbol $f^{i}$ denotes the $i$ th function, we can derive a total of $n^{2}$ partial derivatives.

$$
\frac{\partial y_{i}}{\partial x_{j}}(i=1,2, \cdots, n ; j=1,2, \cdots, n)
$$

We can arrange them into a square matrix, known as the Jacobian matrix and denoted by $\boldsymbol{J}$, and then take its determinant, the result will be what is known as a Jacobian determinant (or a Jacobian, for short), denoted by $|\boldsymbol{J}|$ :

$$
|\boldsymbol{J}|=\left|\frac{\partial\left(y_{1}, y_{2}, \cdots, y_{n}\right)}{\partial\left(x_{1}, x_{2}, \cdots, x_{n}\right)}\right|=\left|\begin{array}{llll}
\frac{\partial y_{1}}{\partial x_{1}} & \frac{\partial y_{1}}{\partial x_{2}} & \cdots & \frac{\partial y_{1}}{\partial x_{n}} \\
\frac{\partial y_{2}}{\partial x_{1}} & \frac{\partial y_{2}}{\partial x_{2}} & \cdots & \frac{\partial y_{2}}{\partial x_{n}} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial y_{n}}{\partial x_{1}} & \frac{\partial y_{n}}{\partial x_{2}} & \cdots & \frac{\partial y_{n}}{\partial x_{n}}
\end{array}\right| .
$$

Example 7.7.1 Consider two functions:

$$
y_{1}=2 x_{1}+3 x_{2} \text {; }
$$

$$
y_{2}=4 x_{1}^{2}+12 x_{1} x_{2}+9 x_{2}^{2} .
$$

Then the Jacobian determinant is

$$
|\boldsymbol{J}|=\left|\begin{array}{cc}
\frac{\partial y_{1}}{\partial x_{1}} & \frac{\partial y_{1}}{\partial x_{2}} \\
\frac{y_{2}}{\partial x_{1}} & \frac{\partial y_{2}}{\partial x_{2}}
\end{array}\right|=\left|\begin{array}{cc}
2 & 3 \\
\left(8 x_{1}+12 x_{2}\right) & \left(12 x_{1}+18 x_{2}\right)
\end{array}\right|=\begin{array}{ll}
0 & \text { for all } \boldsymbol{x} .
\end{array}
$$

A useful test for determining the existence of functional dependence among a set of $n$ functions is the Jacobian determinant. The following theorem provides a formal statement of this test:

Theorem 7.7.1 The $n$ functions $f^{1}, f^{2}, \cdots f^{n}$ are functionally dependent (either linearly or nonlinearly) if and only if the Jacobian determinant $|\boldsymbol{J}|$ is identically zero for all $\boldsymbol{x}=\left(x_{1}, x_{2}, \cdots, x_{n}\right)$.

This theorem implies that if the Jacobian determinant is non-zero for at least one $\boldsymbol{x}$, then the functions $f^{1}, f^{2}, \cdots f^{n}$ are functionally independent at that point. Conversely, if the Jacobian determinant is identically zero, then the functions are functionally dependent for all possible values of $x_{1}, x_{2}, \cdots, x_{n}$.

For the above example, since

$$
|\boldsymbol{J}|=\left|\frac{\partial\left(y_{1}, y_{2}\right)}{\partial\left(x_{1}, x_{2}\right)}\right|=\left(24 x_{1}+36 x_{2}\right)-\left(24 x_{1}+36 x_{2}\right) \equiv 0
$$

for all $x_{1}$ and $x_{2}$, then $y^{1}$ and $y^{2}$ are functionally dependent. In fact, $y_{2}$ is simply $y_{1}$ squared.

Let us now consider the special case of linear functions. We have earlier shown that the rows of the coefficient matrix $\boldsymbol{A}$ of a linear-equation system: $\boldsymbol{A} \boldsymbol{x}=\boldsymbol{d}$, i.e.,

$$
a_{11} x_{1}+a_{12} x_{2}+\cdots+a_{1 n} x_{n}=d_{1}
$$

$$
\begin{gathered}
a_{21} x_{1}+a_{22} x_{2}+\cdots+a_{2 n} x_{n}=d_{2} ; \\
\cdots \\
a_{n 1} x_{1}+a_{n 2} x_{2}+\cdots+a_{n n} x_{n}=d_{n} .
\end{gathered}
$$

We can interpret the fact that the rows of the coefficient matrix $\boldsymbol{A}$ are linearly dependent if and only if $|\boldsymbol{A}|=0$ as a special application of the Jacobian criterion of functional dependence.

To see this, we can treat each equation in $\boldsymbol{A x}=\boldsymbol{d}$ as a separate function of the $n$ variables $x_{1}, x_{2}, \cdots, x_{n}$ and denote these functions by $y_{1}, y_{2}, \cdots, y_{n}$. Then, we have $\partial y_{i} / \partial x_{j}=a_{i j}$. With this in mind, the elements of $|\boldsymbol{J}|$ are precisely the elements of $\boldsymbol{A}$, i.e., $|\boldsymbol{J}|=|\boldsymbol{A}|$. Hence, the Jacobian criterion of functional dependence among $y_{1}, y_{2}, \cdots, y_{n}$ is equivalent to the criterion $|\boldsymbol{A}|=0$ in the present linear case.

126CHAPTER 7. RULES OF DIFFERENTIATION AND THEIR USE IN COMPARATIVE S

## Chapter 8

## Comparative-Static Analysis of General Functions

The study of partial derivatives has enabled us, in the preceding chapter, to handle the simple type of comparative-static problems in which the equilibrium solution of the model can be explicitly stated in reduced form . We note that the definition of the partial derivative requires the absence of any functional relationship among the independent variables. As applied to comparative-static analysis, this means that parameters and/or exogenous variables that appear in the reduced-form solution must be mutually independent.

However, we cannot expect such expediency when, due to the inclusion of general functions in a model, no explicit reduced-form solution can be obtained. In such a case, we must find the comparative-static derivatives directly from the originally given equations in the model. For example, consider a simple national income model with two endogenous variables, $Y$ and $C$ :

$$
\left.Y=C+I_{0}+G_{0} \quad \text { equilibrim condition }\right) ;
$$

$$
C=C\left(Y, T_{0}\right) \quad\left(T_{0}: \text { exogenous taxes }\right),
$$

which reduces to a single equation

$$
Y=C\left(Y, T_{0}\right)+I_{0}+G_{0}
$$

to be solved for $\bar{Y}$. We must, therefore, find the comparative-static derivatives directly from this equation. How might we approach the problem?

Let us assume that there exists an equilibrium solution, denoted by $\bar{Y}$. We can express this equilibrium as a function of the exogenous variables $I_{0}, G_{0}$, and $T_{0}$ :

$$
\bar{Y}=\bar{Y}\left(I_{0}, G_{0}, T_{0}\right)
$$

although we may not be able to determine the exact form of this function. Moreover, in a neighborhood of $\bar{Y}$, we can write the following equality:

$$
\bar{Y} \equiv C\left(\bar{Y}, T_{0}\right)+I_{0}+G_{0},
$$

where $C$ is a function that depends on $\bar{Y}$ and $T_{0}$. Note that since $\bar{Y}$ is a function of $T_{0}$, the two arguments of the $C$ function are not independent. This means that $T_{0}$ can affect $C$ not only directly but also indirectly through its effect on $\bar{Y}$. Therefore, partial differentiation is not appropriate in this case, and we need to use total differentiation instead. Total differentiation can lead us to the concept of total derivative, which allows us to analyze functions where the arguments are not all independent, and study the comparative statics of general function models.

### 8.1 Differentials

The symbol $d y / d x$ has been regarded as a single entity. We shall now reinterpret as a ratio of two quantities, $d y$ and $d x$.

## Differentials and Derivatives

Given a function $y=f(x)$, we can use the difference quotient $\Delta y / \Delta x$ to represent the ratio of change of $y$ with respect to $x$. Since

$$
\begin{equation*}
\Delta y \equiv\left[\frac{\Delta y}{\Delta x}\right] \Delta x \tag{8.1.1}
\end{equation*}
$$

the magnitude of $\Delta y$ can be found, once the $\Delta y / \Delta x$ and the variation $\Delta x$ are known. If we denote the infinitesimal changes in $x$ and $y$, respectively, by $d x$ and $d y$, the identity (8.1) becomes

$$
\begin{equation*}
d y \equiv\left[\frac{d y}{d x}\right] d x \tag{8.1.2}
\end{equation*}
$$

or

$$
\begin{equation*}
d y=f^{\prime}(x) d x \tag{8.1.3}
\end{equation*}
$$

The symbols $d y$ and $d x$ are known as the differentials of $y$ and $x$, respectively.

Dividing the two identities in (8.1.2) throughout by $d x$, we have

$$
\frac{(d y)}{(d x)} \equiv\left(\frac{d y}{d x}\right) .
$$

or

$$
\frac{(d y)}{(d x)} \equiv f^{\prime}(x) .
$$

This result shows that the derivative $d y / d x \equiv f^{\prime}(x)$ can be interpreted as the quotient of two separate differentials $d y$ and $d x$.

On the basis of (8.1.2), once we are given $f^{\prime}(x), d y$ can immediately be written as $f^{\prime}(x) d x$. The derivative $f^{\prime}(x)$ may thus be viewed as a "converter" that serves to convert an infinitesimal change $d x$ into a corresponding
change $d y$.
The following diagram shows the relationship between " $\Delta y$ " and " $d y$ ".


Figure 8.1: Graphical illustration of the relationship between " $\Delta y$ " and " $d y$ ".

$$
\begin{aligned}
\Delta y & \equiv\left[\frac{\Delta y}{\Delta x}\right] \Delta x=\frac{C B}{A C} A C=C B \\
d y & =\left[\frac{d y}{d x}\right] \Delta x=\frac{C D}{A C} A C=C D
\end{aligned}
$$

which differs from $\Delta y$ by an error of $D B$.
Example 8.1.1 Given $y=3 x^{2}+7 x-5$, find $d y$.
Since $f^{\prime}(x)=6 x+7$, the desired differential is

$$
d y=(6 x+7) d x
$$

Remark 8.1.1 The purpose of finding the differential $d y$ is called the differentiation. Recall that we have also used this term as a synonym for derivation. To avoid confusion, the word "differentiation" with the phrase "with respect to $x$ " when we take derivative $d y / d x$.

## Differentials and Point Elasticity

As an illustration of the application of differentials in economics, let us consider the elasticity of a function. For a demand function $Q=f(P)$, for instance, the price elasticity of demand is defined as $(\Delta Q / Q) /(\Delta P / P)$, the ratio of percentage change in quantity demanded and percentage change in price. Now if $\Delta P \rightarrow 0$, the $\Delta P$ and $\Delta Q$ will reduce to the differential $d P$ and $d Q$, and the elasticity becomes

$$
\epsilon_{d} \equiv \frac{d Q / Q}{d P / P}=\frac{d Q / d P}{Q / P}=\frac{\text { marginal demand function }}{\text { average demand function }}
$$

In general, given $y=f(x)$, the point elasticity of $y$ with respect to $x$ as

$$
\epsilon_{y x}=\frac{d y / d x}{y / x}=\frac{\text { marginal function }}{\text { average function }}=\frac{d y}{d x} \times \frac{x}{y} .
$$

Example 8.1.2 Suppose the demand function is given by $Q=100-2 P$. We want to find the price elasticity of demand, $\epsilon_{d}$.

Since $d Q / d P=-2$ and $Q / P=(100-2 P) / P$, so $\epsilon_{d}=(-P) /(50-P)$. Thus the demand is inelastic $\left(\left|\epsilon_{d}\right|<1\right)$ for $0<P<25$, unit elastic ( $\left|\epsilon_{d}\right|=1$ ) for $P=25$, and elastic for $25<P<50$.

### 8.2 Total Differentials

The concept of differentials can be extended to a function of two or more independent variables. Let's consider a savings function,

$$
S=S(Y, i),
$$

where $S$ is savings, $Y$ is national income, and $i$ is the interest rate. If the function is continuous and possesses continuous partial derivatives, we can define the total differential as:

$$
d S=\frac{\partial S}{\partial Y} d Y+\frac{\partial S}{\partial i} d i
$$

which means that the infinitesimal change in $S$ is the sum of the infinitesimal change in $Y$ and the infinitesimal change in $i$.

Remark 8.2.1 If $i$ remains constant, the total differential will reduce to the partial differential:

$$
\frac{\partial S}{\partial Y}=\left(\frac{d S}{d Y}\right)_{i \text { constant }}
$$

Furthermore, general case of a function of $n$ variables $y=f\left(x_{1}, x_{2}, \cdots, x_{n}\right)$, the total differential of this function is given by

$$
d f=\frac{\partial f}{\partial x_{1}} d x_{1}+\frac{\partial f}{\partial x_{2}} d x_{2}+\cdots+\frac{\partial f}{\partial x_{n}} d x_{n}=\sum_{i=1}^{n} f_{i} d x_{i}
$$

in which each term on the right side indicates the amount of change in $y$ resulting from an infinitesimal change in one of $n$ variables.

Similar to the case of one variable, the $n$ partial elasticities can be written as

$$
\epsilon_{f x_{i}}=\frac{\partial f}{\partial x_{i}} \frac{x_{i}}{f}(i=1,2, \cdots, n) .
$$

### 8.3 Rule of Differentials

Let $c$ be constant and $u$ and $v$ be two functions of the variables $x_{1}, x_{2}, \cdots, x_{n}$. The the following rules are valid:

Rule I: $d c=0$;

Rule II: $d\left(c u^{a}\right)=c a u^{a-1} d u$;

Rule III: $d(u \pm v)=d u \pm d v ;$

Rule IV: $d(u v)=v d u+u d v ;$

Rule V: $d(u / v)=1 / v^{2}(v d u-u d v)$.

Example 8.3.1 Find $d y$ of the function $y=5 x_{1}^{2}+3 x_{2}$. There are two ways to find $d y$. One is the straightforward method by finding $\partial f / \partial x_{1}$ and $\partial f / \partial x_{2}$ : $\partial f / \partial x_{1}=10 x_{1}$ and $\partial f / \partial x_{2}=3$, which will then enable us to write

$$
d y=f_{1} d x_{1}+f_{2} d x_{2}=10 x_{1} d x_{1}+3 d x_{2} .
$$

The other way is to use the differential rules given above by letting $u=5 x_{1}^{2}$ and $v=3 x_{2}$;

$$
\begin{aligned}
d y & =d\left(5 x_{1}^{2}\right)+d\left(3 x_{2}\right)(\text { by rule III }) \\
& =10 x_{1} d x_{1}+3 d x_{2}(\text { by rule II }) .
\end{aligned}
$$

Example 8.3.2 Find $d y$ of the function $y=3 x_{1}^{2}+x_{1} x_{2}^{2}$. Since $f_{1}=6 x_{1}+x_{2}^{2}$ and $f_{2}=2 x_{1} x_{2}$, the desired differential is

$$
d y=\left(6 x_{1}+x_{2}^{2}\right) d x_{1}+2 x_{1} x_{2} d x_{2}
$$

By applying the given rules, the same result can be arrived at

$$
\begin{aligned}
d y & =d\left(3 x_{1}^{2}\right)+d\left(x_{1} x_{2}^{2}\right) \\
& =6 x_{1} d x_{1}+x_{2}^{2} d x_{1}+2 x_{1} x_{2} d x_{2} \\
& =\left(6 x_{1}+x_{2}^{2}\right) d x_{1}+2 x_{1} x_{2} d x_{2} .
\end{aligned}
$$

Example 8.3.3 For the function

$$
\begin{gathered}
y=\frac{x_{1}+x_{2}}{2 x_{1}^{2}} \\
f_{1}=\frac{-\left(x_{1}+2 x_{2}\right)}{2 x_{1}^{3}} \text { and } f_{2}=\frac{1}{2 x_{1}^{2}},
\end{gathered}
$$

then

$$
d y=\frac{-\left(x_{1}+2 x_{2}\right)}{2 x_{1}^{3}} d x_{1}+\frac{1}{2 x_{1}^{2}} d x_{2} .
$$

The same result can also be obtained by applying the given rules:

$$
\begin{aligned}
d y & =\frac{1}{4 x_{1}^{4}}\left[2 x_{1}^{2} d\left(x_{1}+x_{2}\right)-\left(x_{1}+x_{2}\right) d\left(2 x_{1}^{2}\right)\right] \quad[\text { by rule } \mathrm{V}] \\
& =\frac{1}{4 x_{1}^{4}}\left[2 x_{1}^{2}\left(d x_{1}+d x_{2}\right)-\left(x_{1}+x_{2}\right) 4 x_{1} d x_{1}\right] \\
& =\frac{1}{4 x_{1}^{4}}\left[-2 x_{1}\left(x_{1}+2 x_{2}\right) d x_{1}+2 x_{1}^{2} d x_{2}\right] \\
& =-\frac{x_{1}+2 x_{2}}{2 x_{1}^{3}} d x_{1}+\frac{1}{2 x_{1}^{2}} d x_{2} .
\end{aligned}
$$

For the case of more than two functions, we have:

Rule VI: $d(u \pm v \pm w)=d u \pm d v \pm d w ;$

Rule VII: $d(u v w)=v w d u+u w d v+u v d w$.

### 8.4 Total Derivatives

Consider a function

$$
y=f(x, w) \text { with } x=g(w)
$$

Here, the variable $w$ can affect $y$ through two channels: (1) indirectly, via the function $g$ and then $f$, and (2) directly, via the function $f$. Unlike a partial derivative, the total derivative takes into account both channels and allows $x$ to change with $w$. It expresses the overall effect of changing $w$ on $y$, taking into account both direct and indirect effects.

While the partial derivative $f_{w}$ can be used to express the direct effect alone, it does not capture the indirect effect through $x$. In contrast, the total derivative incorporates both channels of effect and provides a more comprehensive understanding of the relationship between $y$ and $w$.

To get the total derivative, we first get the total differential

$$
d y=f_{x} d x+f_{w} d w
$$

Dividing both sides of this equation by $d w$ leads to the total derivative:

$$
\begin{aligned}
\frac{d y}{d w} & =f_{x} \frac{d x}{d w}+f_{w} \frac{d w}{d w} \\
& =\frac{\partial y}{\partial x} \frac{d x}{d w}+\frac{\partial y}{\partial w} .
\end{aligned}
$$

Example 8.4.1 Find the $d y / d w$, given the function

$$
\begin{gathered}
y=f(x, w)=3 x-w^{2} \text { with } x=g(w)=2 w^{2}+w+4 . \\
\frac{d y}{d w}=f_{x} \frac{d x}{d w}+f_{w}=3(4 w+1)-2 w=10 w+3 .
\end{gathered}
$$

As a check, we may substitute the function $g$ into $f$, to get

$$
y=3\left(2 w^{2}+w+4\right)-w^{2}=5 w^{2}+3 w+12
$$

which is now a function of $w$ alone. Then, we also have

$$
\frac{d y}{d w}=10 w+3
$$

and thus we have the identical answer.

Example 8.4.2 Suppose that a utility function is given by

$$
U=U(s, c),
$$

where $c$ is the amount of coffee consumed and $s$ is the amount of sugar consumed, and another function $s=g(c)$ indicates the complementarity between these two goods. Then we can find the marginal utility of coffee given by

$$
M U_{c}=\frac{d U}{d c}=\frac{\partial U}{\partial s} g^{\prime}(c)+\frac{\partial U}{\partial c} .
$$

Through the inverse function rule for $c=g^{-1}(s)$, we can also find the marginal utility of sugar given by

$$
\begin{aligned}
M U_{s}=\frac{d U}{d s} & =\frac{\partial U}{\partial c} \frac{d c}{d s}+\frac{\partial U}{\partial s} \\
& =\frac{\partial U}{\partial c} \frac{1}{g^{\prime}(c)}+\frac{\partial U}{\partial s}
\end{aligned}
$$

The marginal rate of substitution of coffee for sugar $M R S_{c s}$ is given by

$$
M R S_{c s}=\frac{M U_{c}}{M U_{s}}=\frac{\frac{\partial U}{\partial s} g^{\prime}(c)+\frac{\partial U}{\partial c}}{\frac{\partial U}{\partial c} \frac{1}{g^{\prime}(c)}+\frac{\partial U}{\partial s}}=g^{\prime}(c)\left[\frac{\frac{\partial U}{\partial s} g^{\prime}(c)+\frac{\partial U}{\partial c}}{\frac{\partial U}{\partial s} g^{\prime}(c)+\frac{\partial U}{\partial c}}\right]=g^{\prime}(c),
$$

which is not strange since $M R S_{c s}$ measures the rate of changes of $c$ and $s$
that is given by $g^{\prime}(c)$.

## A Variation on the Theorem

For a function

$$
y=f\left(x_{1}, x_{2}, w\right)
$$

with $x_{1}=g(w)$ and $x_{2}=h(w)$, the total derivative of $y$ is given by

$$
\frac{d y}{d w}=\frac{\partial f}{\partial x_{1}} \frac{d x_{1}}{d w}+\frac{\partial f}{\partial x_{2}} \frac{d x_{2}}{d w}+\frac{\partial f}{\partial w} .
$$

Example 8.4.3 Let a production function be

$$
Q=Q(K, L, t),
$$

where $K$ is the capital input, $L$ is the labor input, and $t$ is the time which indicates that the production can shift over time in reflection of technological change. Since capital and labor can also change over time, we may write

$$
K=K(t) \text { and } L=L(t)
$$

Thus the rate of output with respect to time can be denote as

$$
\frac{d Q}{d t}=\frac{\partial Q}{\partial K} \frac{d K}{d t}+\frac{\partial Q}{\partial L} \frac{d L}{d t}+\frac{\partial Q}{\partial t} .
$$

## Another Variation on the Theme

Now if a function is given,

$$
y=f\left(x_{1}, x_{2}, u, v\right)
$$

with $x_{1}=g(u, v)$ and $x_{2}=h(u, v)$, we can find the total derivative of $y$ with respect to $u$ (while $v$ is held constant). Since

$$
d y=\frac{\partial f}{\partial x_{1}} d x_{1}+\frac{\partial f}{\partial x_{2}} d x_{2}+\frac{\partial f}{\partial u} d u+\frac{\partial f}{\partial v} d v
$$

dividing both sides of the above equation by $d u$, we have

$$
\begin{aligned}
\frac{d y}{d u} & =\frac{\partial y}{\partial x_{1}} \frac{d x_{1}}{d u}+\frac{\partial y}{\partial x_{2}} \frac{d x_{2}}{d u}+\frac{\partial y}{\partial u} \frac{d u}{d u}+\frac{\partial y}{\partial v} \frac{d v}{d u} \\
& =\frac{\partial y}{\partial x_{1}} \frac{d x_{1}}{d u}+\frac{\partial y}{\partial x_{2}} \frac{d x_{2}}{d u}+\frac{\partial y}{\partial u}\left[\frac{d v}{d u}=0 \text { since } v \text { is constant }\right] .
\end{aligned}
$$

Since $v$ is held constant, the above is the partial total derivative, denoted by the section symbol $\S$, we redenote the above equation by the following notation:

$$
\frac{\S y}{\S u}=\frac{\partial y}{\partial x_{1}} \frac{\partial x_{1}}{\partial u}+\frac{\partial y}{\partial x_{2}} \frac{\partial x_{2}}{\partial u}+\frac{\partial y}{\partial u} .
$$



$$
z=3 x^{2}-2 y^{4}+5 u v^{2},
$$

where

$$
x=u-v^{2}+4 .
$$

and

$$
y=8 u^{3} v+v^{2}+1
$$

By applying the above formula on the partial total derivative, we have

$$
\begin{aligned}
\frac{\S z}{\S u} & =\frac{\partial z}{\partial x} \frac{\partial x}{\partial u}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial u}+\frac{\partial z}{\partial u} \\
& =6 x \times 1-8 y^{3} \times 24 u^{2} v+5 v^{2} \\
& =6 x-196 y^{3} u^{2} v+5 v^{2} .
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\S z}{\S v} & =\frac{\partial z}{\partial x} \frac{\partial x}{\partial v}+\frac{\partial z}{\partial y} \frac{\partial y}{\partial v}+\frac{\partial z}{\partial v} \\
& =6 x \times-2 v-8 y^{3} \times\left(8 u^{3}+2 v\right)+10 u v \\
& =-12 x v-8 y^{3}\left(8 u^{3}+2 v\right)+10 u v .
\end{aligned}
$$

Remark 8.4.1 In the cases we have discussed, the formulas for the total derivative can be viewed as applications of the chain rule, also known as the composite function rule. Furthermore, the chain of derivatives is not restricted to only two links. The concept of the total derivative can be extended to situations where there are three or more links in the composite function.

### 8.5 Implicit Function Theorem

The concept of total differentials can also enable us to find the derivatives of so-called "implicit functions." As a result, we can still conduc$t$ comparative-static analysis for general functions without obtaining an explicit function.

## Implicit Functions

A function given in the form of $y=f\left(x_{1}, x_{2}, \cdots, x_{n}\right)$ is known as an explicit function because the variable $y$ is explicitly expressed as a function of $\boldsymbol{x}=\left(x_{1}, x_{2}, \cdots, x_{n}\right)$. However, in many cases, $y$ is not an explicit function of $\boldsymbol{x}$. Instead, the relationship between $y$ and $\boldsymbol{x}$ is given in the form of

$$
F(y, \boldsymbol{x})=0 .
$$

Such an equation may define an implicit function $y=f(\boldsymbol{x})$. Note that
an explicit function $y=f\left(x_{1}, x_{2}, \cdots, x_{n}\right)$ can always be transformed into an equation

$$
F\left(y, x_{1}, x_{2}, \cdots, x_{n}\right) \equiv y-f\left(x_{1}, x_{2}, \cdots, x_{n}\right)=0 .
$$

However, the reverse transformation is not always possible, which introduces some uncertainty.

To address this uncertainty, we can impose certain conditions under which we ensure that a given equation $F(y, \boldsymbol{x})=0$ indeed defines an implicit function $y=f(\boldsymbol{x})$. The Implicit-Function Theorem provides such conditions.

Theorem 8.5.1 (Implicit-Function Theorem) Suppose that $F(y, \boldsymbol{x}): \mathbb{R}^{n+1} \rightarrow$ $\mathbb{R}$ is a continuously differentiable function, and $F\left(y_{0}, \boldsymbol{x}_{0}\right)=0$ for some point $\left(y_{0}, \boldsymbol{x}_{0}\right) \in \mathbb{R}^{n+1}$. If $F_{y}\left(y_{0}, \boldsymbol{x}_{0}\right) \neq 0$, then there is a neighborhood $N\left(\boldsymbol{x}^{0}\right)$ of $\left(\boldsymbol{x}_{0}\right)$ such that:
(1) A function $y=f(\boldsymbol{x})$ can be defined implicitly on $N\left(\boldsymbol{x}_{0}\right)$, satisfying:

$$
F(y(\boldsymbol{x}), \boldsymbol{x})=0 .
$$

(2) The function $y=f(\boldsymbol{x})$ is continuous on $N\left(\boldsymbol{x}_{0}\right)$.
(3) The function $y=f(\boldsymbol{x})$ has continuous partial derivatives on $N\left(\boldsymbol{x}_{0}\right)$, which are given by:

$$
\frac{\partial y}{\partial x_{i}}=-\frac{F_{x_{i}}}{F_{y}}, \quad i=1, \ldots, n
$$

The Implicit-Function Theorem provides an important tool for studying implicit functions, which are essential in many areas of mathematics, including calculus, differential equations, and geometry. It enables us to convert an implicit equation into an explicitly defined function, which allows us to analyze and solve problems more easily.

## Derivatives of Implicit Functions

Differentiating $F$, we have $d F=0$, or

$$
F_{y} d y+F_{1} d x_{1}+\cdots+F_{n} d x_{n}=0 .
$$

Suppose that only $y$ and $x_{1}$ are allowed to vary. Then the above equation reduce to $F_{y} d y+F_{1} d x_{1}=0$. Thus

$$
\left.\frac{d y}{d x_{1}}\right|_{\text {other variable constant }} \equiv \frac{\partial y}{\partial x_{1}}=-\frac{F_{1}}{F_{y}} .
$$

In the simple case where the given equation is $F(y, x)=0$, the rule gives

$$
\frac{d y}{d x}=-\frac{F_{x}}{F_{y}}
$$

Example 8.5.1 Suppose $y-3 x^{4}=0$. Then $\frac{d y}{d x}=-\frac{F_{x}}{F_{y}}=-\frac{-12 x^{3}}{1}=12 x^{3}$.
In this particular case, we can easily solve the given equation for $y$, to get $y=3 x^{4}$ so that $d y / d x=12 x^{3}$.

Example 8.5.2 $F(x, y)=x^{2}+y^{2}-9=0$, which gives us a circle with radius 3. Thus,

$$
\frac{d y}{d x}=-\frac{F_{x}}{F_{y}}=-\frac{2 x}{2 y}=-\frac{x}{y},(y \neq 0) .
$$

Example 8.5.3 $F(y, x, w)=y^{3} x^{2}+w^{3}+y x w-3=0$, we have

$$
\frac{\partial y}{\partial x}=-\frac{F_{x}}{F_{y}}=-\frac{2 y^{3} x+y w}{3 y^{2} x^{2}+x w} .
$$

In particular, at point $(1,1,1), \frac{\partial y}{\partial x}=-3 / 4$.

Example 8.5.4 Suppose that the transformation frontier $F(Q, K, L)=0$ of a production technology implicitly defines a production function $Q=$
$f(K, L)$. Then we can find $M P_{L}$ (marginal product of labor) and $M P_{K}$ (marginal product of capital) as follows:

$$
\begin{aligned}
M P_{K} & \equiv \frac{\partial Q}{\partial K}=-\frac{F_{K}}{F_{Q}} \\
M P_{L} & \equiv \frac{\partial Q}{\partial L}=-\frac{F_{L}}{F_{Q}} .
\end{aligned}
$$

In particular, we can also find the $M R T S_{L K}$ (marginal rate of technical substitution) which is given by

$$
M R T S_{L K} \equiv\left|\frac{\partial K}{\partial L}\right|=\frac{F_{L}}{F_{K}} .
$$

## Extension to the Simultaneous-Equation Case

The Implicit-Function Theorem can be extended to the case with any number of dependent variables $\left(y_{1}, y_{2}, \cdots, y_{n}\right)$.

Consider a set of simultaneous equations.

$$
\begin{gathered}
F^{1}\left(y_{1}, y_{2}, \cdots, y_{n} ; x_{1}, x_{2}, \cdots, x_{m}\right)=0 \\
F^{2}\left(y_{1}, y_{2}, \cdots, y_{n} ; x_{1}, x_{2}, \cdots, x_{m}\right)=0 \\
\cdots \\
F^{n}\left(y_{1}, y_{2}, \cdots, y_{n} ; x_{1}, x_{2}, \cdots, x_{m}\right)=0 .
\end{gathered}
$$

Suppose that $F^{1}, F^{2}, \cdots, F^{n}$ are continuously differentiable. Taking differentials on both side of the equation system, we then have

$$
\begin{aligned}
& \frac{\partial F^{1}}{\partial y_{1}} d y_{1}+\frac{\partial F^{1}}{\partial y_{2}} d y_{2}+\cdots+\frac{\partial F^{1}}{\partial y_{n}} d y_{n}=-\left[\frac{\partial F^{1}}{\partial x_{1}} d x_{1}+\frac{\partial F^{1}}{\partial x_{2}} d x_{2}+\cdots+\frac{\partial F^{1}}{\partial x_{m}} d x_{m}\right] ; \\
& \frac{\partial F^{2}}{\partial y_{1}} d y_{1}+\frac{\partial F^{2}}{\partial y_{2}} d y_{2}+\cdots+\frac{\partial F^{2}}{\partial y_{n}} d y_{n}=-\left[\frac{\partial F^{2}}{\partial x_{1}} d x_{1}+\frac{\partial F^{2}}{\partial x_{2}} d x_{2}+\cdots+\frac{\partial F^{2}}{\partial x_{m}} d x_{m}\right] ;
\end{aligned}
$$

$\frac{\partial F^{n}}{\partial y_{1}} d y_{1}+\frac{\partial F^{n}}{\partial y_{2}} d y_{2}+\cdots+\frac{\partial F^{1}}{\partial y_{n}} d y_{n}=-\left[\frac{\partial F^{n}}{\partial x_{1}} d x_{1}+\frac{\partial F^{n}}{\partial x_{2}} d x_{2}+\cdots+\frac{\partial F^{n}}{\partial x_{m}} d x_{m}\right]$.

Or in matrix form,

$$
\left[\begin{array}{cccc}
\frac{\partial F^{1}}{\partial y_{1}} & \frac{\partial F^{1}}{\partial y_{2}} & \cdots & \frac{\partial F^{1}}{\partial y_{n}}  \tag{8.5.4}\\
\frac{\partial F^{2}}{\partial y_{1}} & \frac{\partial F^{2}}{\partial y_{2}} & \cdots & \frac{\partial F^{2}}{\partial y_{n}} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial F^{n}}{\partial y_{1}} & \frac{\partial F^{n}}{\partial y_{2}} & \cdots & \frac{\partial F^{n}}{\partial y_{n}}
\end{array}\right]\left[\begin{array}{cccc}
\frac{\partial F^{1}}{\partial x_{1}} & \frac{\partial F^{1}}{\partial x_{2}} & \cdots & \frac{\partial F^{1}}{\partial x_{m}} \\
\frac{\partial F^{2}}{\partial x_{1}} & \frac{\partial F^{2}}{\partial x_{2}} & \cdots & \frac{\partial F^{2}}{\partial x_{m}} \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial F^{n}}{\partial x_{1}} & \frac{\partial F^{n}}{\partial x_{2}} & \cdots & \frac{\partial F^{n}}{\partial x_{m}}
\end{array}\right]\left[\begin{array}{c}
d x_{1} \\
d y_{2} \\
\cdots \\
d x_{m}
\end{array}\right] .
$$

Now suppose that the following Jacobian determinant is nonzero:

$$
|\boldsymbol{J}|=\left|\frac{\partial\left(F^{1}, F^{2}, \cdots, F^{n}\right)}{\partial\left(y_{1}, y_{2}, \cdots, y_{n}\right)}\right|=\left|\begin{array}{cccc}
\frac{\partial F^{1}}{\partial y_{1}} & \frac{\partial F^{1}}{\partial y_{2}} & \cdots & \frac{\partial F^{1}}{\partial y_{n}} \\
\frac{\partial F^{2}}{\partial y_{1}} & \frac{\partial F^{2}}{\partial y_{2}} & \cdots & \frac{\partial F^{2}}{\partial y_{n}} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial F^{n}}{\partial y_{1}} & \frac{\partial F^{n}}{\partial y_{2}} & \cdots & \frac{\partial F^{n}}{\partial y_{n}}
\end{array}\right| \neq 0 .
$$

Then, we can obtain total differentials $d \boldsymbol{y}=\left(d y_{1}, d y_{2}, \ldots, d y_{n}\right)^{\prime}$ by inverting $J$.

$$
d \boldsymbol{y}=\boldsymbol{J}^{-1} \boldsymbol{F}_{x} d x
$$

where

$$
F_{x}=\left[\begin{array}{cccc}
\frac{\partial F^{1}}{\partial x_{1}} & \frac{\partial F^{1}}{\partial x_{2}} & \cdots & \frac{\partial F^{1}}{\partial x_{m}} \\
\frac{\partial F^{2}}{\partial x_{1}} & \frac{\partial F^{2}}{\partial x_{2}} & \cdots & \frac{\partial F^{2}}{\partial x_{m}} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial F^{n}}{\partial x_{1}} & \frac{\partial F^{n}}{\partial x_{2}} & \cdots & \frac{\partial F^{n}}{\partial x_{m}}
\end{array}\right] .
$$

If we want to obtain partial derivatives with respect to $x_{i}(i=1,2, \ldots, m)$, we can do so by letting $d x_{k}=0$ for $k \neq i$ and dividing both sides of (8.5.4)
by $d x_{i}$. Then, we have the following equation:

$$
\left[\begin{array}{cccc}
\frac{\partial F^{1}}{\partial y_{1}} & \frac{\partial F^{1}}{\partial y_{2}} & \cdots & \frac{\partial F^{1}}{\partial y_{n}} \\
\frac{\partial F^{2}}{\partial y_{1}} & \frac{\partial F^{2}}{\partial y_{2}} & \cdots & \frac{\partial F^{2}}{\partial y_{n}} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial F^{n}}{\partial y_{1}} & \frac{\partial F^{n}}{\partial y_{2}} & \cdots & \frac{\partial F^{n}}{\partial y_{n}}
\end{array}\right]\left[\begin{array}{c}
\frac{\partial y_{1}}{\partial x_{i}} \\
\frac{\partial y_{2}}{\partial x_{i}} \\
\cdots \\
\frac{\partial x_{i}}{\partial x_{i}}
\end{array}\right]=-\left[\begin{array}{c}
\frac{\partial F^{2}}{\partial x_{i}} \\
\cdots \\
\frac{\partial F^{n}}{\partial x_{i}}
\end{array}\right] .
$$

Then, by Cramer's rule, we have

$$
\frac{\partial y_{j}}{\partial x_{i}}=\frac{\left|\boldsymbol{J}_{j}^{i}\right|}{|\boldsymbol{J}|}(j=1,2, \cdots, n ; i=1,2, \cdots, m)
$$

where $\left|\boldsymbol{J}_{j}^{i}\right|$ is obtained by replacing the $j$ th column of $|\boldsymbol{J}|$ with

$$
\boldsymbol{F}_{x_{i}}=\left[\frac{\partial F^{1}}{\partial x_{i}}, \frac{\partial F^{2}}{\partial x_{i}}, \cdots, \frac{\partial F^{n}}{\partial x_{i}}\right]^{\prime}
$$

Of course, we can find these derivatives by inversing the Jacobian matrix $J$ :

$$
\left[\begin{array}{l}
\frac{\partial y_{1}}{\partial x_{i}} \\
\frac{\partial y_{2}}{\partial x_{i}} \\
\cdots \\
\frac{\partial y_{n}}{\partial x_{i}}
\end{array}\right]=-\left[\begin{array}{llll}
\frac{\partial F^{1}}{\partial y_{1}} & \frac{\partial F^{1}}{\partial y_{2}} & \cdots & \frac{\partial F^{1}}{\partial y_{n}} \\
\frac{\partial F^{2}}{\partial y_{1}} & \frac{\partial F^{2}}{\partial y_{2}} & \cdots & \frac{\partial F^{2}}{\partial y_{n}} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial F^{n}}{\partial y_{1}} & \frac{\partial F^{n}}{\partial y_{2}} & \cdots & \frac{\partial F^{n}}{\partial y_{n}}
\end{array}\right]^{-1}\left[\begin{array}{l}
\frac{\partial F^{1}}{\partial x_{i}} \\
\frac{\partial F^{2}}{\partial x_{i}} \\
\cdots \\
\frac{\partial F^{n}}{\partial x_{i}}
\end{array}\right] .
$$

In the compact notation,

$$
\frac{\partial \boldsymbol{y}}{\partial x_{i}}=-\boldsymbol{J}^{-1} \boldsymbol{F}_{x_{i}} .
$$

Example 8.5.5 Let the national-income model be rewritten in the form:

$$
\begin{gathered}
Y-C-I_{0}-G_{0}=0 \\
C-\alpha-\beta(Y-T)=0
\end{gathered}
$$

$$
T-\gamma-\delta Y=0
$$

Then

$$
|\boldsymbol{J}|=\left|\begin{array}{lll}
\frac{\partial F^{1}}{\partial Y} & \frac{\partial F^{1}}{\partial C} & \frac{\partial F^{1}}{\partial T} \\
\frac{\partial F^{2}}{\partial Y} & \frac{\partial F^{2}}{\partial C} & \frac{\partial F^{2}}{\partial T} \\
\frac{\partial F^{3}}{\partial Y} & \frac{\partial F^{3}}{\partial C} & \frac{\partial F^{3}}{\partial T}
\end{array}\right|=\left|\begin{array}{ccc}
1 & -1 & 0 \\
-\beta & 1 & \beta \\
-\delta & 0 & 1
\end{array}\right|=1-\beta+\beta \delta .
$$

Suppose that all exogenous variables and parameters are fixed except $G_{0}$. Then we have

$$
\left[\begin{array}{ccc}
1 & -1 & 0 \\
-\beta & 1 & \beta \\
-\delta & 0 & 1
\end{array}\right]\left[\begin{array}{c}
\frac{\partial \bar{Y}}{\partial G_{0}} \\
\frac{\partial \bar{C}}{\partial G_{0}} \\
\frac{\partial \bar{T}}{\partial G_{0}}
\end{array}\right]=\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right] .
$$

We can solve the above equation for, say, $\partial \bar{Y} / \partial G_{0}$ which comes out to be

$$
\frac{\partial \bar{Y}}{\partial G_{0}}=\frac{\left|\begin{array}{ccc}
1 & -1 & 0 \\
0 & 1 & \beta \\
0 & 0 & 1
\end{array}\right|}{|J|}=\frac{1}{1-\beta+\beta \delta}
$$

### 8.6 Comparative Statics of General-Function Models

Consider a single-commodity market model:

$$
\begin{aligned}
& Q_{d}=Q_{s}, \quad[\text { equilibrium condition }] ; \\
& Q_{d}=D\left(P, Y_{0}\right), \quad\left[\partial D / \partial P<0 ; \partial D / \partial Y_{0}>0\right] ; \\
& Q_{s}=S(P), \quad[d S / d P>0],
\end{aligned}
$$

where $Y_{0}$ is an exogenously determined income. From this model, we can obtain a single equation, namely, excess demand:

$$
F\left(P, Y_{0}\right)=D\left(P, Y_{0}\right)-S(P)=0
$$

Although this equation cannot be solved explicitly for the equilibrium price $\bar{P}$, by the implicit-function theorem, we know that there exists the equilibrium price $\bar{P}$ that is the function of $Y_{0}$ :

$$
\bar{P}=\bar{P}\left(Y_{0}\right),
$$

such that

$$
F\left(\bar{P}, Y_{0}\right)=D\left(\bar{P}, Y_{0}\right)-S(\bar{P})=0
$$

It then requires only a straight application of the implicit-function rule to produce the comparative-static derivative, $d \bar{P} / d Y_{0}$ :

$$
\frac{d \bar{P}}{d Y_{0}}=-\frac{\partial F / \partial Y_{0}}{\partial F / \partial P}=-\frac{\partial D / \partial Y_{0}}{\partial D / \partial P-d S / d P}>0
$$

Since $\bar{Q}=S(\bar{P})$, we have

$$
\frac{d \bar{Q}}{d Y_{0}}=\frac{d S}{d P} \frac{d \bar{P}}{d Y_{0}}>0 .
$$

### 8.7 Matrix Derivatives

Matrix derivatives play an important role in economic analysis, especially in econometrics. If $\boldsymbol{A}$ is a $n \times n$ non-singular matrix, the derivative of its determinant with respect to $\boldsymbol{A}$ is given by

$$
\frac{\partial}{\partial \boldsymbol{A}}|\boldsymbol{A}|=\left[\boldsymbol{C}_{i j}\right]
$$

where $\left[\boldsymbol{C}_{i j}\right]$ is the matrix of cofactors of $\boldsymbol{A}$.

## Some Useful Formulas

Let $\boldsymbol{a}, \boldsymbol{b}$ be $k \times 1$ vectors and $\boldsymbol{M}$ be a $k \times k$ matrix. Then we have:

$$
\begin{aligned}
& \frac{d \boldsymbol{a}^{\prime} \boldsymbol{b}}{d \boldsymbol{b}}=\boldsymbol{a} \\
& \frac{d \boldsymbol{b}^{\prime} \boldsymbol{a}}{d \boldsymbol{b}}=\boldsymbol{a} \\
& \frac{d \boldsymbol{M} \boldsymbol{b}}{d \boldsymbol{b}}=\boldsymbol{M}^{\prime} ; \\
& \frac{d \boldsymbol{b}^{\prime} \boldsymbol{M} \boldsymbol{b}}{d b}=\left(\boldsymbol{M}+\boldsymbol{M}^{\prime}\right) \boldsymbol{b} .
\end{aligned}
$$

148CHAPTER 8. COMPARATIVE-STATIC ANALYSIS OF GENERAL FUNCTIONS

## Chapter 9

## Optimization: Maxima and Minima of a Function of One <br> Variable

The optimization problem lies at the core of economics. The assumption that individuals act rationally by maximizing their personal interests is fundamental to economic analysis and practice. This assumption leads to the study of goal equilibrium, in which the equilibrium state is defined as the optimal position for a given economic unit, and the said unit will deliberately strive for attainment of that equilibrium. Classical techniques for locating optimal positions, those using differential calculus, are the primary focus of our attention.

### 9.1 Optimal and Extreme Values

Economics is a science of choice. When an economic project is to be carried out, there are normally a number of alternative ways of accomplishing it. One (or more) of these alternatives will be more desirable than others from
the standpoint of some criterion. The essence of the optimization problem is to choose the most desirable alternative.

The most common criterion for choice among alternatives in economics is the goal of maximizing something (e.g., utility, profit) or minimizing something (e.g., cost). Such maximization and minimization problems fall under the general heading of optimization. From a purely mathematical point of view, the collective term for maximum and minimum is the more matter-of-fact designation extremum, meaning an extreme value.

In formulating an optimization problem, the first order of business is to delineate an objective function, in which the dependent variable represents the objects whose magnitudes the economic unit in question can pick and choose. The independent variables are referred to as choice variables.

Consider a general-form objective function

$$
y=f(x) .
$$



Figure 9.1: The extremum for various functions: (a) constant function; (b) monotonic function, (3) non-monotonic function.

Figure 9.1 depicts three specific cases of functions. Points E and F in
(c) are referred to as relative (or local) extremum, as they represent an extremum in some neighborhood of the point only. Our discussion will mainly focus on the search for relative extrema. An absolute (or global) maximum must be either a relative maximum or one of the ends of the function. Thus, if we know all the relative maxima, it is necessary only to select the largest of these and compare it with the end points in order to determine the absolute maximum. Hereafter, the extreme values considered will be relative or local ones, unless indicated otherwise.

### 9.2 Existence of Extremum for Continuous Functions

Let $X$ be the domain of a function $f$. We give the following concepts:

Definition 9.2.1 (Local Optimum) Let $f(x)$ be a continuous function defined on $X \subseteq \mathbb{R}$ (or, in the general case, of $R^{n}$ ). It is said to have a local or relative maximum (resp. minimum) at $x_{0} \in X$ if there is in a neighborhood $U$ of $x_{0}$ such that $f(x) \leq f\left(x_{0}\right)$ (resp. $\left.f(x) \geq f\left(x_{0}\right)\right)$ for all $x \in U$.

Definition 9.2.2 (Global Optimum) If $f\left(x^{*}\right) \geqq f(x)$ (resp. $f\left(x^{*}\right)>f(x)$ ) for all $x \in X$, then the function is said to have global (unique) maximum at $x^{*}$; if $f\left(x^{*}\right) \leqq f(x)$ (resp. $f\left(x^{*}\right)<f(x)$ ) for all $x \in X$, then the function is said to have global (unique) minimum at $x^{*}$.

A classical conclusion about global optimization is the so-called Weierstrass theorem.

Proposition 9.2.1 (Weierstrass's Theorem) Suppose that $f$ is continuous on a closed and bounded subset $X$ of $R^{1}$ (or, in the general case, of $R^{n}$ ). Then, $f$ reaches its maximum and minimum in $X$, i.e. there exist points $m, M \in X$ such
that $f(m) \leq f(x) \leq f(M)$ for all $x \in X$. Moreover, the set of maximal (resp. minimal) points is compact.

In order to determine whether a function has an extreme point, we can use the method of finding extreme values by the differential method. Generally, there are two types of necessary conditions for the interior extreme point, i.e., the first and second-order necessary conditions.

### 9.3 First-Derivative Test for Relative Maximum and Minimum

Given a function $y=f(x)$, the first derivative $f^{\prime}(x)$ plays a key role in determining its extreme values. For smooth functions, an interior relative extreme value can only occur where $x$ is known as a stationary point, i.e., where $f^{\prime}(x)=0$. Such points are called "stationary" because at these points, the function may "stop" increasing or decreasing. A stationary point is also known as a critical point.

A turning point is a point at which the derivative changes sign. A turning point may be either a relative maximum or a relative minimum. If the function is differentiable, then a turning point is a stationary point; however, not all stationary points are turning points, such as inflection points (e.g., $y=x^{3}$ ).

Thus, $f^{\prime}(x)=0$ is a necessary (but not sufficient) condition for a relative extremum (either maximum or minimum). We summarize this necessary condition for extremum in the following proposition.

## Proposition 9.3.1 (Fermat's Theorem: Necessary Condition for Extremum)

 Suppose that $f(x)$ is differentiable on $X$ and has a local extremum (minimum or maximum) at an interior point $x_{0} \in X$. Then $f^{\prime}\left(x_{0}\right)=0$.

Figure 9.2: The first derivative test: (a) $f^{\prime}\left(x_{0}\right)$ does not exist; and (b) $f^{\prime}\left(x_{0}\right)=0$.

Note that if the first derivative vanishes at some point, it does not imply that $f$ possesses an extremum at that point. For example, $f=x^{3}$ has a stationary point at $x=0$ but does not have an extremum at that point.

We have some useful results about stationary points.
Proposition 9.3.2 (Rolle's Theorem) Suppose that $f$ is continuous in $[a, b]$, differentiable on $(a, b)$, and $f(a)=f(b)$. Then there exists a point $c \in(a, b)$ such that $f^{\prime}(c)=0$.

From Rolle's Theorem, we can prove the well-known Mean-Value Theorem, also called Lagrange's Theorem.

Proposition 9.3.3 (Mean Value Theorem or Lagrange's Theorem) Suppose that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Then there exists a point $c \in(a, b)$ such that $f^{\prime}(c)=\frac{f(b)-f(a)}{b-a}$.

Proof. Let $g(x)=f(x)-\frac{f(b)-f(a)}{b-a} x$. Then, $g$ is continuous in $[a, b]$, differentiable on $(a, b)$ and $g(a)=g(b)$. Thus, by Rolle's Theorem, there exists one point $c \in(a, b)$ such that $g^{\prime}(c)=0$, and therefore $f^{\prime}(c)=\frac{f(b)-f(a)}{b-a}$.


Figure 9.3: The Mean Value Theorem implies that there exists some $c$ in the interval $(a, b)$ such that the secant joining the endpoints of the interval $[a, b]$ is parallel to the tangent at $c$.

The above Mean Value Theorem is also true for multivariate $\boldsymbol{x}$. If function $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is differentiable, then there is $\boldsymbol{z}=t \boldsymbol{x}+(1-t) \boldsymbol{y}$ with $0 \leq t \leq 1$, such that

$$
f(\boldsymbol{y})=f(\boldsymbol{x})+D f(\boldsymbol{z})(\mathbf{y}-\mathbf{x})
$$

where

$$
D f(\boldsymbol{x})=\left[\frac{\partial f(\boldsymbol{x})}{\partial x_{1}}, \frac{\partial f(\boldsymbol{x})}{\partial x_{2}}, \cdots, \frac{\partial f(\boldsymbol{x})}{\partial x_{n}}\right]
$$

A variation of the above Mean Value Theorem is in the form of integral calculus:

Theorem 9.3.1 (Mean Value Theorem of Integral Calculus) Suppose that $f$ : $[a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$. Then there exists a number $c \in(a, b)$ such that

$$
\int_{a}^{b} f(x) d x=f(c)(b-a) .
$$

Proof. Let $F(x)=\int_{a}^{x} f(t) d t$. Since $f:[a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$, $F(x)$ is continuous and differentiable on $(a, b)$. Then, by the Mean Value

### 9.3. FIRST-DERIVATIVE TEST FOR RELATIVE MAXIMUM AND MINIMUM155

Theorem, there is $c \in(a, b)$ such that

$$
\frac{F(b)-F(a)}{b-a}=F^{\prime}(c)=f(c) .
$$

Therefore, we have

$$
\int_{a}^{b} f(x) d x=f(c)(b-a) .
$$

The second variation of the mean-value theorem is the generalized mean-value theorem:

## Proposition 9.3.4 (Cauchy's Theorem or the Generalized Mean Value Theorem)

Suppose that $f$ and $g$ are continuous in $[a, b]$ and differentiable in $(a, b)$. Then there exists a point $c \in(a, b)$ such that

$$
(f(b)-f(a)) g^{\prime}(c)=(g(b)-g(a)) f^{\prime}(c) .
$$

Proof. The case where $g(a)=g(b)$ is easy. So, assume that $g(a) \neq g(b)$. Define

$$
h(x)=f(x)-\frac{f(b)-f(a)}{g(b)-g(a)} g(x) .
$$

Then, applying the Mean-Value Theorem to $h(x)$ on $[a, b]$ gives

$$
h(b)-h(a)=h^{\prime}(c)(b-a)=0,
$$

where $c$ is a point in $(a, b)$. Therefore,

$$
f(b)-f(a)-\frac{f(b)-f(a)}{g(b)-g(a)}(g(b)-g(a))=0,
$$

which implies

$$
(f(b)-f(a)) g^{\prime}(c)=(g(b)-g(a)) f^{\prime}(c) .
$$

This completes the proof.
To determine whether a function has a local maximum or minimum, we can use the following proposition on the first-derivative test relative extremum.

Proposition 9.3.5 (First-Derivative Test Relative Extremum) Suppose that $f^{\prime}\left(x_{0}\right)=0$. Then the value of the function at $x_{0}, f\left(x_{0}\right)$, is
(a) a relative maximum if $f^{\prime}(x)$ changes its sign from positive to negative from the immediate left of the point $x_{0}$ to its immediate right;
(b) a relative minimum if $f^{\prime}(x)$ changes its sign from negative to positive from the immediate left of the point $x_{0}$ to its immediate right;
(c) an inflection (not extreme) point if $f^{\prime}(x)$ has the same sign on some neighborhood.

Thus, a relative maximum or minimum must be a turning point.
Example 9.3.1 Consider the function $y=(x-1)^{3}$.
We have $f^{\prime}(x)=3(x-1)^{2}$, which gives $f^{\prime}(1)=0$. However, $x=1$ is not an extreme point because $f^{\prime}(x)$ does not change sign around $x=1$.

Example 9.3.2 Consider the function $y=f(x)=x^{3}-12 x^{2}+36 x+8$.
We have $f^{\prime}(x)=3 x^{2}-24 x+36=3(x-2)(x-6)$, which leads to $f^{\prime}(x)=0$ when $x=2$ or $x=6$. We also have $f^{\prime}(x)>0$ for $x<2$ and $f^{\prime}(x)<0$ for $x>2$ when $x$ is sufficiently close to 2 . Therefore, $x=2$ is a local maximal point and the corresponding maximum value of the function is $f(2)=40$. Similarly, we can verify that $x=6$ is a local minimal point and the corresponding minimum value is $f(6)=8$ as the minimum.

Example 9.3.3 Find the relative extremum of the average-cost function

$$
A C=f(Q)=Q^{2}-5 Q+8
$$

Since $f^{\prime}(2.5)=0, f^{\prime}(Q)<0$ for $Q<2.5$, and $f^{\prime}(Q)>0$ for $Q>2.5$, the average cost reaches its minimum at $\bar{Q}=2.5$.

### 9.4 Second and Higher Derivatives

Since the first derivative $f^{\prime}(x)$ of a function $y=f(x)$ is also a function of $x$, we can consider the derivative of $f^{\prime}(x)$, which is called the second derivative. Similarly, we can find derivatives of even higher orders. These will enable us to develop alternative criteria for locating the relative extrema of a function.

The second derivative of the function $f$ is denoted by $f^{\prime \prime}(x)$ or $d^{2} y / d x^{2}$. If the second derivative $f^{\prime \prime}(x)$ exists for all $x$ values, $f(x)$ is said to be twice differentiable. If, in addition, $f^{\prime \prime}(x)$ is continuous, $f(x)$ is said to be twice continuously differentiable.

The higher-order derivatives of $f(x)$ can be obtained and symbolized similarly to the second derivative

$$
f^{\prime \prime \prime}(x), f^{(4)}(x), \cdots, f^{(n)}(x),
$$

or

$$
\frac{d^{3} y}{d x^{3}}, \frac{d^{4} y}{d x^{4}}, \cdots, \frac{d^{n} y}{d x^{n}}
$$

Remark 9.4.1 $d^{n} y / d x^{n}$ can be also written as $\left(d^{n} / d x^{n}\right) y$, where the $d^{n} / d x^{n}$ part serves as an operator symbol instructing us to take the $\mathbf{n}$-th derivative with respect to $x$.

Example 9.4.1 $y=f(x)=4 x^{4}-x^{3}+17 x^{2}+3 x-1$.

Then

$$
\begin{aligned}
& f^{\prime}(x)=16 x^{3}-3 x^{2}+34 x+3 ; \\
& f^{\prime \prime}(x)=48 x^{2}-6 x+34 ; \\
& f^{\prime \prime \prime}(x)=96 x-6 ; \\
& f^{(4)}(x)=96 ; \\
& f^{(5)}(x)=0 .
\end{aligned}
$$

Example 9.4.2 Find the first four derivatives of the function

$$
y=g(x)=\frac{x}{1+x} \quad(x \neq-1) .
$$

$$
\begin{aligned}
& g^{\prime}(x)=(1+x)^{-2} ; \\
& g^{\prime \prime}(x)=-2(1+x)^{-3} ; \\
& g^{\prime \prime \prime}(x)=6(1+x)^{-4} \\
& g^{(4)}(x)=-24(1+x)^{-5} .
\end{aligned}
$$

Remark 9.4.2 A negative second derivative is consistently reflected in an inverse U-shaped curve; a positive second derivative is reflected in an Ushaped curve.

### 9.5 Second-Derivative Test

Recall the meaning of the first and second derivatives of a function $f$. The sign of the first derivative tells us whether the function is increasing ( $f^{\prime}>$ 0 ) or decreasing $\left(f^{\prime}<0\right)$, while the sign of the second derivative tells us whether the slope of the function is increasing $\left(f^{\prime \prime}>0\right)$ or decreasing $\left(f^{\prime \prime}<\right.$ 0 ). This gives us insight into how to verify the existence of a maximum
or minimum at a stationary point. We have the following result on the second-derivative test for relative extremum.

Proposition 9.5.1 (Second-Derivative Test for Relative Extremum) Suppose that $f^{\prime}\left(x_{0}\right)=0$. Then, the value of the function at $x_{0}, f\left(x_{0}\right)$, will be
(a) a relative maximum if $f^{\prime \prime}\left(x_{0}\right)<0$;
(b) a relative minimum if $f^{\prime \prime}\left(x_{0}\right)>0$.

This test is generally more convenient to use than the first-derivative test, as it does not require us to check the derivative sign to both the left and right of $x$.

Example 9.5.1 Consider $y=f(x)=4 x^{2}-x$.
Since $f^{\prime}(x)=8 x-1$ and $f^{\prime \prime}(x)=8$, we know $f(x)$ reaches its minimum at $\bar{x}=1 / 8$. Indeed, since the function plots as a U-shaped curve, the relative minimum is also the absolute minimum.

Example 9.5.2 Consider $y=g(x)=x^{3}-3 x^{2}+2$.
$y^{\prime}=g^{\prime}(x)=3 x^{2}-6 x$ and $y^{\prime \prime}=6 x-6$. Setting $g^{\prime}(x)=0$, we obtain two stationary points $\bar{x}_{1}=0$ and $\bar{x}_{2}=2$, which in turn yield the two extreme values $g(0)=2$ (a maximum because $\left.g^{\prime \prime}(0)=-6<0\right)$ and $g(2)=-2(\mathrm{a}$ minimum because $\left.g^{\prime \prime}(2)=6>0\right)$.

Remark 9.5.1 Note that when $f^{\prime}\left(x_{0}\right)=0, f^{\prime \prime}\left(x_{0}\right)<0\left(\right.$ resp. $\left.f^{\prime \prime}\left(x_{0}\right)>0\right)$ is a sufficient condition for a relative maximum (resp. minimum), but not a necessary condition. However, the condition $f^{\prime \prime}\left(x_{0}\right) \leq 0\left(f^{\prime \prime}\left(x_{0}\right) \geq 0\right)$ is necessary (although not sufficient) for a relative maximum (resp. minimum).

## Condition for Profit Maximization

Let $R=R(Q)$ be the total-revenue function and let $C=C(Q)$ be the total-cost function, where $Q$ is the level of output. The profit function is then given by

$$
\pi=\pi(Q)=R(Q)-C(Q)
$$

To find the profit-maximizing output level, we need to find $\bar{Q}$ such that

$$
\pi^{\prime}(\bar{Q})=R^{\prime}(\bar{Q})-C^{\prime}(\bar{Q})
$$

or

$$
R^{\prime}(\bar{Q})=C^{\prime}(\bar{Q}), \text { or } M R(\bar{Q})=M C(\bar{Q}) .
$$

To ensure the first-order condition leads to a maximum, we require

$$
\frac{d^{2} \pi}{d Q} \equiv \pi^{\prime \prime}(\bar{Q})-C^{\prime \prime}(\bar{Q})<0 .
$$

Economically, this means that at the output level $\bar{Q}$, the marginal revenue $M R$ is equal to the marginal cost $M C$. If the rate of change of $M R$ is less than the rate of change of $M C$ at $\bar{Q}$, then increasing output would decrease profit, and therefore $\bar{Q}$ would maximize profit.

Example 9.5.3 Let $R(Q)=1200 Q-2 Q^{2}$ and $C(Q)=Q^{3}-61.25 Q^{2}+$ $1528.5 Q+2000$. Then the profit function is

$$
\pi(Q)=-Q^{3}+59.2 Q^{2}-328.5 Q-2000
$$

Setting $\pi^{\prime}(Q)=-3 Q^{2}+118.5 Q-328.5=0$, we have $\bar{Q}_{1}=3$ and $\bar{Q}_{2}=36.5$. Since $\pi^{\prime \prime}(3)=-18+118.5=100.5>0$ and $\pi^{\prime \prime}(36.5)=-219+$ $118.5=-100.5<0$, so the profit-maximizing output is $\bar{Q}=36.5$. At this output level, the marginal revenue and marginal cost are both 56.5 , and
increasing or decreasing output would result in a lower profit.

### 9.6 Taylor Series

This section discusses the expansion of a function $y=f(x)$ into the Taylor series around any point $x=x_{0}$. Expanding a function $y=f(x)$ around a point $x_{0}$ involves transforming the function into a polynomial form, where the coefficients of the different terms are expressed in terms of the derivative values $f^{\prime}\left(x_{0}\right), f^{\prime \prime}\left(x_{0}\right)$, etc., all evaluated at the point of expansion $x_{0}$.

Proposition 9.6.1 (Taylor's Theorem) Given an arbitrary function $\phi(x)$, if we know the values of $\phi\left(x_{0}\right), \phi^{\prime}\left(x_{0}\right), \phi^{\prime \prime}\left(x_{0}\right)$, etc., then this function can be expanded around the point $x_{0}$ as follows:

$$
\begin{aligned}
\phi(x) & =\phi\left(x_{0}\right)+\phi^{\prime}\left(x_{0}\right)\left(x-x_{0}\right)+\frac{1}{2!} \phi^{\prime \prime}\left(x_{0}\right)\left(x-x_{0}\right)^{2}+\cdots+\frac{1}{n!} \phi^{(n)}\left(x_{0}\right)\left(x-x_{0}\right)^{n}+R_{n} \\
& \equiv P_{n}+R_{n}
\end{aligned}
$$

where $P_{n}$ represents the nth-degree polynomial and $R_{n}$ denotes a remainder which can be expressed in the so-called Lagrange form of the remainder:

$$
R_{n}=\frac{1}{(n+1)!} \phi^{(n+1)}\left(x_{\lambda}\right)\left(x-x_{0}\right)^{n+1},
$$

where $\xi$ is some value between $x_{0}$ and $x$.

In other words, the Taylor series of a function $\phi(x)$ around $x_{0}$ is a representation of the function as an infinite sum of terms, each of which is a power of $x-x_{0}$ multiplied by a coefficient involving the derivatives of $\phi(x)$ evaluated at $x_{0}$. The Lagrange form of the remainder gives an estimate of the error in the approximation, with the size of the error decreasing as $n$ in-
creases. Taylor series expansions are useful in many areas of mathematics and science, including calculus, physics, and engineering.


Figure 9.4: The graphic representation of the Taylor's Theorem reduces to the mean-value theorem when $n=0$.

Remark 9.6.1 When $n=0$, the Taylor's Theorem reduces to the meanvalue theorem that we discussed in Section 9.3:

$$
\phi(x)=P_{0}+R_{0}=\phi\left(x_{0}\right)+\phi^{\prime}\left(x_{\lambda}\right)\left(x-x_{0}\right),
$$

or

$$
\phi(x)-\phi\left(x_{0}\right)=\phi^{\prime}\left(x_{\lambda}\right)\left(x-x_{0}\right),
$$

which states that the difference between the value of the function $\phi$ at $x_{0}$ and at any other $x$ value can be expressed as the product of the difference $\left(x-x_{0}\right)$ and $\phi^{\prime}\left(x_{\lambda}\right)$ with $x_{\lambda}$ being some point between $x$ and $x_{0}$.

Remark 9.6.2 If $x_{0}=0$, then Taylor series reduce to the so-called Maclau-

## rin series:

$\phi(x)=\phi(0)+\phi^{\prime}(0) x+\frac{1}{2!} \phi^{\prime \prime}(0) x^{2}+\cdots+\frac{1}{n!} \phi^{(n)}(0) x^{n}+\frac{1}{(n+1)!} \phi^{(n+1)}\left(x_{\lambda}\right) x^{n+1}$,
where $x_{\lambda}$ is a point between 0 and $x$.
Example 9.6.1 Expand the function

$$
\phi(x)=\frac{1}{1+x}
$$

around the point $x_{0}=1$, with $n=4$. Since $\phi(1)=\frac{1}{2}$ and

$$
\begin{aligned}
& \phi^{\prime}(x)=-(1+x)^{-2}, \phi^{\prime}(1)=-1 / 4 ; \\
& \phi^{\prime \prime}(x)=2(1+x)^{-3}, \phi^{\prime \prime}(1)=1 / 4 ; \\
& \phi^{(3)}(x)=-6(1+x)^{-4}, \phi^{(3)}(1)=-3 / 8 ; \\
& \phi^{(4)}(x)=24(1+x)^{-5}, \phi^{(4)}(1)=3 / 4,
\end{aligned}
$$

we obtain the following Taylor series:

$$
\begin{aligned}
\phi(x)= & \phi(1) \\
& +\phi^{\prime}(1)(x-1)+\frac{1}{2!} \phi^{\prime \prime}(1)(x-1)^{2}+\frac{1}{3!} \phi^{\prime \prime \prime}(1)(x-1)^{3} \\
& +\frac{1}{n!} \phi^{(4)}(1)(x-1)^{4}+R_{4} \\
= & 1 / 2-1 / 4(x-1)+1 / 8(x-1)^{2}-1 / 16(x-1)^{3} \\
& +1 / 32(x-1)^{4}+\frac{\phi^{(n+1)}\left(x_{\lambda}\right)}{5!}(x-1)^{5} .
\end{aligned}
$$

### 9.7 Nth-Derivative Test

A relative extremum of the function $f$ can be defined as follows:
A function $f(x)$ attains a relative maximum (resp. minimum) value at $x_{0}$ if $f(x)-f\left(x_{0}\right)$ is nonpositive (resp. nonnegative) for values of $x$ in some neighborhood of $x_{0}$.

Assume that $f(x)$ has finite, continuous derivatives up to the $N$-th order at $x=x_{0}$. Then the function can be expanded around $x=x_{0}$ as a Taylor series:

$$
\begin{aligned}
f(x)-f\left(x_{0}\right)= & f^{\prime}\left(x_{0}\right)\left(x-x_{0}\right)+\frac{1}{2!} f^{\prime \prime}\left(x_{0}\right)\left(x-x_{0}\right)^{2}+\cdots \\
& +\frac{1}{(N-1)!} f^{(N-1)}\left(x_{0}\right)\left(x-x_{0}\right)^{N-1}+\frac{1}{N!} f^{(N)}\left(x_{\lambda}\right)\left(x-x_{0}\right)^{N} .
\end{aligned}
$$

Now if

$$
f^{\prime}\left(x_{0}\right)=f^{\prime \prime}\left(x_{0}\right)=\ldots=f^{(N-1)}\left(x_{0}\right)=0
$$

and

$$
f^{(N)}\left(x_{\lambda}\right)\left(x-x_{0}\right)^{N} \neq 0,
$$

the above equation reduces to

$$
\begin{equation*}
f(x)-f\left(x_{0}\right)=\frac{1}{N!} f^{(N)}\left(x_{\lambda}\right)\left(x-x_{0}\right)^{N} \neq 0 . \tag{9.7.1}
\end{equation*}
$$

Then, we have the following proposition.
Proposition 9.7.1 (Nth-Derivative Test) Suppose that $f^{\prime}\left(x_{0}\right)=0$, and the first nonzero derivative value at $x_{0}$ encountered in successive derivation is that of the Nth derivative, $f^{(N)}\left(x_{0}\right) \neq 0$. Then the stationary value $f\left(x_{0}\right)$ will be
(a) a relative maximum if $N$ is an even number and $f^{(N)}\left(x_{0}\right)<0$;
(b) a relative minimum if $N$ is an even number and $f^{(N)}\left(x_{0}\right)>0$;
(c) an inflection point if $N$ is odd.

Example 9.7.1 Consider the function $y=(7-x)^{4}$.
Since $f^{\prime}(7)=4(7-7)^{3}=0, f^{\prime \prime}(7)=12(7-7)^{2}=0, f^{\prime \prime \prime}(7)=24(7-7)=0$, $f^{(n)}(7)=24>0$, so $x=7$ is a relative minimum with $f(7)=0$.

## Chapter 10

## Exponential and Logarithmic Functions

Exponential functions and logarithmic functions are important mathematical tools with diverse applications in economics, finance, science, and engineering. In this chapter, we will discuss some fundamental properties of these functions and their derivatives.

### 10.1 Exponential Functions

An exponential function is a function of the form:

$$
y=f(t)=b^{t}, \quad(b>0, b \neq 1)
$$

where $b$ is a fixed base of the exponent $t$. The base $b$ determines the rate at which the function grows or decays. When $b>1$, the function grows exponentially, and when $0<b<1$, the function decays exponentially. Its generalized version has the form:

$$
y=a b^{c t} .
$$

Exponential functions have several important properties:

1. The domain of an exponential function is all real numbers.
2. The range of an exponential function is $(0, \infty)$.
3. Exponential functions are one-to-one and onto on their respective domains and ranges.

Remark 10.1.1 $y=a b^{c t}=a\left(b^{c}\right)^{t}$. Thus we can consider $b^{c}$ as a base of exponent $t$. It changes exponent from $c t$ to $t$ and changes base $b$ to $b^{c}$.

The exponential function $y=e^{t}$, where $e \approx 2.71828$ is the mathematical constant known as Euler's number, is a special case of the exponential function, known as the natural exponential function, which can be alternatively denoted as

$$
y=a \exp (r t)
$$

Remark 10.1.2 It can be proved that $e$ may be defined as the limit:

$$
e \equiv \lim _{n \rightarrow \infty} f(n)=\lim _{n \rightarrow \infty}\left(1+\frac{1}{n}\right)^{n}
$$

### 10.2 Logarithmic Functions

For the exponential function $y=b^{t}$ and the natural exponential function $y=e^{t}$, taking the $\log$ of $y$ to the base $b$ (denote $\operatorname{byy}^{\log } \log _{b} y$ ) and the base $e$ (denoted by $\log _{e} y$ ) respectively, we obtain the logarithmic function.

$$
t=\log _{b} y
$$

and

$$
t=\log _{e} y \equiv \ln y
$$

For example, we know that $4^{2}=16$. So we can write $\log _{4} 16=2$.

Since $y=b^{t} \Longleftrightarrow t=\log _{b} y$, we can write

$$
b^{\log _{b} y}=b^{t}=y .
$$

The following rules are familiar to us:

## Rules:

(a) $\ln (u v)=\ln u+\ln v \quad$ (log of product);
(b) $\ln (u / v)=\ln u-\ln v \quad$ (log of quotient);
(c) $\ln u^{a}=a \ln u \quad$ (log of power);
(d) $\log _{b} u=\left(\log _{b} e\right)\left(\log _{e} u\right)=\left(\log _{b} e\right)(\ln u) \quad($ conversion of $\log$ base);
(e) $\log _{b} e=1 /\left(\log _{e} b\right)=1 / \ln b \quad$ (inversion of $\log$ base).

## Properties of Log:

(a) $\log y_{1}=\log y_{2}$ iff $y_{1}=y_{2}$;
(b) $\log y_{1}>\log y_{2}$ iff $y_{1}>y_{2}$;
(c) $0<y<1$ iff $\log y<0$;
(d) $y=1$ iff $\log y=0$;
(e) $\log y \rightarrow \infty$ as $y \rightarrow \infty$;
(f) $\log y \rightarrow-\infty$ as $y \rightarrow 0$.

Remark 10.2.1 $t=\log _{b} y$ and $t=\ln y$ are the respective inverse functions of the exponential functions $y=b^{t}$ and $y=e^{t}$.

### 10.3 Derivatives of Exponential and Logarithmic Functions

## The Basic Rule:

(a) $\frac{d \ln t}{d t}=\frac{1}{t}$;
(b) $\frac{d e^{t}}{d t}=e^{t}$;
(c) $\frac{d e^{f(t)}}{d t}=f^{\prime}(t) e^{f(t)}$;
(d) $\frac{d}{d t} \ln f(t)=\frac{f^{\prime}(t)}{f(t)}$.

Example 10.3.1 The following are examples to find derivatives:
(a) Let $y=e^{r t}$. Then $d y / d t=r e^{r t}$;
(b) Let $y=e^{-t}$. Then $d y / d t=-e^{-t}$;
(c) Let $y=\ln a t$. Then $d y / d t=a / a t=1 / t$;
(d) Let $y=\ln t^{c}$. Since $y=\ln t^{c}=c \ln t$, so $d y / d t=c(1 / t)$;
(e) Let $y=t^{3} \ln t^{2}$. Then $d y / d t=3 t^{2} \ln t^{2}+2 t^{3} / t=2 t^{2}(1+3 \ln t)$.

## The Case of Base $b$

(a) $\frac{d b^{t}}{d t}=b^{t} \ln b$;
(b) $\frac{d}{d t} \log _{b} t=\frac{1}{t \ln b}$;
(c) $\frac{d}{d t} b^{f(t)}=f^{\prime}(t) b^{f(t)} \ln b$;
(d) $\frac{d}{d t} \log _{b} f(t)=\frac{f^{\prime}(t)}{f(t)} \frac{1}{\ln b}$.

Proof of (a). Since $b^{t}=e^{\ln b^{t}}=e^{t \ln b}$, then $(d / d t) b^{t}=(d / d t) e^{t \ln b}=$ $(\ln b)\left(e^{t \ln b}\right)=b^{t} \ln b$.

Proof of (b). Since

$$
\begin{gathered}
\log _{b} t=\left(\log _{b} e\right)\left(\log _{e} t\right)=(1 / \ln b) \ln t \\
(d / d t)\left(\log _{b} t\right)=(d / d t)[(1 / \ln b) \ln t]=(1 / \ln b)(1 / t)
\end{gathered}
$$

Example 10.3.2 (a) Let $y=12^{1-t}$. Then $\frac{d y}{d t}=\frac{d(1-t)}{d t} 12^{1-t} \ln 12=-12^{1-t} \ln 12$

## An Application

Example 10.3.3 To find $d y / d x$ from $y=x^{a} e^{k x-c}$, we take the natural log of both sides:

$$
\ln y=a \ln x+k x-c
$$

Differentiating both sides with respect to $x$, we get

$$
\frac{1}{y} \frac{d y}{d x}=\frac{a}{x}+k
$$

Multiplying both sides by $y$, we get

$$
\frac{d y}{d x}=(a / x+k) y=(a / x+k) x^{a} e^{k x-c}
$$

Example 10.3.4 Use the above technical method, we can similarly find the derivative of $y=\phi(x)^{\psi(x)}$, which cannot be fund out by any other rule.

Taking the natural $\log$ of both sides, we have

$$
\ln y=\psi(x) \ln \phi(x)
$$

Differentiating both sides with respect to $x$ leads to

$$
\frac{1}{y} \frac{d y}{d x}=\psi^{\prime}(x) \ln \phi(x)+\frac{\psi(x) \phi^{\prime}(x)}{\phi(x)}
$$

170 CHAPTER 10. EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Multiplying both sides by $y$, we get

$$
\frac{d y}{d x}=\phi(x)^{\psi(x)}\left[\psi^{\prime}(x) \ln \phi(x)+\frac{\psi(x) \phi^{\prime}(x)}{\phi(x)}\right] .
$$

## Chapter 11

## Optimization for a Function of Two or More Variables

This chapter discusses finding the extreme values of an objective function that involves two or more choice variables. Our focus will be heavily on relative extrema, and unless otherwise specified, the extrema referred to are relative.

### 11.1 The Differential Version of Optimization Conditions

In this section, we demonstrate the equivalence between the derivative version of first and second-order conditions and their differential counterparts.

We first consider the function $z=f(x)$ with only one variable. Recall that the differential of $z=f(x)$ is

$$
d z=f^{\prime}(x) d x
$$

Since $f^{\prime}(x)=0$ is the necessary condition for extreme values, it implies that $d z=0$ is also necessary. This first-order condition requires that $d z=0$ as $x$ is varied. In such a context, with $d x \neq 0, d z=0$ if and only if $f^{\prime}(x)=0$.

What about the sufficient conditions in terms of second-order differentials?

Differentiating $d z=f^{\prime}(x) d x$, we get

$$
\begin{aligned}
d^{2} z & \equiv d(d z)=d\left[f^{\prime}(x) d x\right] \\
& =d\left[f^{\prime}(x)\right] d x \\
& =f^{\prime \prime}(x) d x^{2} .
\end{aligned}
$$

Note that the symbols $d^{2} z$ and $d x^{2}$ are fundamentally different. $d^{2} z$ means the second-order differential of $z$; whereas $d x^{2}$ means the squaring of the first-order differential $d x$.

Thus, from the above equation, we have $d^{2} z<0$ (resp. $d^{2} z>0$ ) if and only if $f^{\prime \prime}(x)<0$ (resp. $f^{\prime \prime}(x)>0$ ). Therefore, the second-order sufficient condition for maximum (resp. minimum) of $z=f(x)$ is $d^{2} z<0$ (resp. $\left.d^{2} z>0\right)$.

### 11.2 Extreme Values of a Function of Two Variables

For a function of one variable, an extreme value is represented graphically by the peak of a hill or the bottom of a valley in a two-dimensional graph. With two choice variables, the graph of the function $z=f(x, y)$ becomes a surface in three-dimensional space, and while the extreme values are still associated with peaks and valleys, they are now identified by looking at the critical points of the function.


Figure 11.1: Graphical illustrations of extrema of a function with two choice variables: (a) $A$ is a maximum; and (b) $B$ is a minimum.

### 11.2.1 First-Order Condition

For a function $z=f(x, y)$, the first-order necessary condition for an extremum involves setting the total differential $d z$ to zero for arbitrary values of $d x$ and $d y$ : an extremum must be reached at a stationary point where $z$ is constant for arbitrary infinitesimal changes in $x$ and $y$.

In the present two-variable case, the total differential is given by

$$
d z=f_{x} d x+f_{y} d y
$$

Thus, the equivalent derivative version of the first-order condition $d z=0$ is

$$
f_{x}=f_{y}=0
$$

or equivalently,

$$
\partial f / \partial x=\partial f / \partial y=0
$$

As in the earlier discussion, the first-order condition is necessary, but not sufficient to identify an extremum. To develop a sufficient condition, we must look at the second-order partial derivatives.

### 11.2.2 Second-Order Partial Derivatives

From the function $z=f(x, y)$, we can obtain two first-order partial derivatives, $f_{x}$ and $f_{y}$. Since $f_{x}$ and $f_{y}$ are themselves functions of $x$ and $y$, we can find second-order partial derivatives:

$$
\begin{aligned}
f_{x x} & \equiv \frac{\partial}{\partial x} f_{x} \text { or } \frac{\partial^{2} z}{\partial x^{2}} \equiv \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \\
f_{y y} & \equiv \frac{\partial}{\partial y} f_{y} \text { or } \frac{\partial^{2} z}{\partial y^{2}} \equiv \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) \\
f_{x y} & \equiv \frac{\partial^{2} z}{\partial x \partial y} \equiv \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \\
f_{y x} & \equiv \frac{\partial^{2} z}{\partial y \partial x} \equiv \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)
\end{aligned}
$$

The last two are known as cross (or mixed) partial derivatives.

Theorem 11.2.1 (Schwarz's Theorem or Young's Theorem) If at least one of the two partials is continuous, then

$$
\frac{\partial^{2} f}{\partial x_{j} \partial x_{i}}=\frac{\partial^{2} f}{\partial x_{i} \partial x_{j}}, i, j=1,2, \cdots, n
$$

Remark 11.2.1 Although $f_{x y}$ and $f_{y x}$ have been separately defined, they will be identical to each other, according to Young's theorem, as long as the two cross-partial derivatives are both continuous. In fact, this theorem also applies to functions of three or more variables. Given $z=g(u, v, w)$, for instance, the mixed partial derivatives will be characterized by $g_{u v}=g_{v u}$, $g_{u w}=g_{w u}$, etc., provided these partial derivatives are continuous.

Example 11.2.1 Find all second-order partial derivatives of

$$
z=x^{3}+5 x y-y^{2} .
$$

The first partial derivatives of this function are:

$$
f_{x}=3 x^{2}+5 y \text { and } f_{y}=5 x-2 y
$$

Thus, $f_{x x}=6 x, f_{y x}=5$, and $f_{y y}=-2$. As expected, $f_{y x}=f_{x y}$.

Example 11.2.2 For $z=x^{2} e^{-y}$, its first partial derivatives are

$$
f_{x}=2 x e^{-y} \text { and } f_{y}=-x^{2} e^{-y}
$$

Again, $f_{y x}=f_{x y}$.

## Second-Order Total Differentials

From the first total differential

$$
d z=f_{x} d x+f_{y} d y,
$$

we can obtain the second-order total differential $d^{2} z$ :

$$
\begin{aligned}
d^{2} z & \equiv d(d z)=\frac{\partial(d z)}{\partial x} d x+\frac{\partial(d z)}{\partial y} d y \\
& =\frac{\partial}{\partial x}\left(f_{x} d x+f_{y} d y\right) d x+\frac{\partial}{\partial y}\left(f_{x} d x+f_{y} d y\right) d y \\
& =\left[f_{x x} d x+f_{x y} d y\right] d x+\left[f_{y x} d x+f_{y y} d y\right] d y \\
& =f_{x x} d x^{2}+f_{x y} d y d x+f_{y x} d x d y+f_{y y} d y^{2} \\
& =f_{x x} d x^{2}+2 f_{x y} d x d y+f_{y y} d y^{2}\left[\text { if } f_{x y}=f_{y x}\right] .
\end{aligned}
$$

We know that if $f(x, y)$ satisfy the conditions of Schwarz's theorem, we have $f_{x y}=f_{y x}$.

Example 11.2.3 Given $z=x^{3}+5 x y-y^{2}$, find $d z$ and $d z^{2}$.

$$
\begin{aligned}
d z & =f_{x} d x+f_{y} d y \\
& =\left(3 x^{2}+5 y\right) d x+(5 x-2 y) d y . \\
d^{2} z & =f_{x x} d x^{2}+2 f_{x y} d x d y+f_{y y} d y^{2} \\
& =6 x d x^{2}+10 d x d y-2 d y^{2} .
\end{aligned}
$$

Note that the second-order total differential can be written in matrix form

$$
\begin{aligned}
d^{2} z & =f_{x x} d x^{2}+2 f_{x y} d x d y+f_{y y} d y^{2} \\
& =[d x, d y]\left[\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right]\left[\begin{array}{l}
d x \\
d y
\end{array}\right]
\end{aligned}
$$

for the function $z=f(x, y)$, where the matrix

$$
\boldsymbol{H}=\left[\begin{array}{ll}
f_{x x} & f_{x y} \\
f_{y x} & f_{y y}
\end{array}\right]
$$

is called the Hessian matrix (or simply a Hessian).
Then, by the discussion on quadratic forms in Chapter 5, we have
(a) $d^{2} z$ is positive definite iff $f_{x x}>0$ and $|\boldsymbol{H}|=f_{x x} f_{y y}-\left(f_{x y}\right)^{2}>$ 0;
(b) $d^{2} z$ is negative definite iff $f_{x x}<0$ and $|\boldsymbol{H}|=f_{x x} f_{y y}-\left(f_{x y}\right)^{2}>$ 0.

From the inequality $f_{x x} f_{y y}-\left(f_{x y}\right)^{2}>0$, it implies that $f_{x x}$ and $f_{y y}$ are required to take the same sign.

Example 11.2.4 Give $f_{x x}=-2, f_{x y}=1$, and $f_{y y}=-1$ at a certain point on a function $z=f(x, y)$, does $d^{2} z$ have a definite sign at that point regardless of the values of $d x$ and $d y$ ? The Hessian determinant is in this case

$$
\left|\begin{array}{cc}
-2 & 1 \\
1 & -1
\end{array}\right|
$$

with the order of the leading principal minors $\left|\boldsymbol{H}_{1}\right|=-2<0$ and

$$
\left|\boldsymbol{H}_{2}\right|=\left|\begin{array}{cc}
-2 & 1 \\
1 & -1
\end{array}\right|=2-1=1>0
$$

Thus $d^{2} z$ is negative definite.

Example 11.2.5 Give $f_{x x}=-2, f_{x y}=1$, and $f_{y y}=-1$ at a certain point on a function $z=f(x, y)$, does $d^{2} z$ have a definite sign at that point regardless of the values of $d x$ and $d y$ ? The Hessian determinant is in this case

$$
\left|\begin{array}{cc}
-2 & 1 \\
1 & -1
\end{array}\right|
$$

with the principal minors $\left|\boldsymbol{H}_{1}\right|=-2<0$ and

$$
\left|\boldsymbol{H}_{2}\right|=\left|\begin{array}{cc}
-2 & 1 \\
1 & -1
\end{array}\right|=2-1=1>0
$$

Thus $d^{2} z$ is negative definite.

For operational convenience, second-order differential conditions can be translated into equivalent conditions on second-order derivatives. The actual translation would require knowledge of quadratic forms, which has already been discussed in Chapter 5 .

## Second-Order Sufficient Condition for Extremum

Using the concept of $d^{2} z$, we have the following:
(a) For maximum of $z=f(x, y)$ : $d^{2} z<0$ for any values of $d x$ and $d y$, not both zero, which is equivalent to:

$$
f_{x x}<0, f_{y y}<0, \text { and } f_{x x} f_{y y}>\left(f_{x y}\right)^{2}
$$

(b) For minimum of $z=f(x, y): d^{2} z>0$ for any values of $d x$ and $d y$, not both zero, which is equivalent to:

$$
f_{x x}>0, f_{y y}>0, \text { and } f_{x x} f_{y y}>\left(f_{x y}\right)^{2}
$$

Remark 11.2.2 When $f_{x x} f_{y y}>\left(f_{x y}\right)^{2}, f_{x x}$ and $f_{y y}$ must have the same sign, i.e., $f_{x x}>0($ resp. $<0)$ implies that $f_{y y}>0($ or $<0)$, otherwise $f_{x x} f_{y y}>$ $\left(f_{x y}\right)^{2}$ cannot be true. Thus, we do not need to check if $f_{y y}>0$ (resp. $<0$ ) when we verify if a stationary point is an extremum.

Therefore, from the above first- and second-order conditions, we obtain the following proposition for relative extrema.

Proposition 11.2.1 (Conditions for Extremum) Suppose that $z=f(x, y)$ is twice continuously differentiable. Then, we have:

## Conditions for Maximum:

(1) $f_{x}=f_{y}=0$ (necessary condition);
(2) $\left|\boldsymbol{H}_{1}\right|=f_{x x}<0$ and $\left|\boldsymbol{H}_{2}\right|=f_{x x} f_{y y}>\left(f_{x y}\right)^{2}$.

Conditions for Minimum:
(1) $f_{x}=f_{y}=0$ (necessary condition);
(2) $\left|\boldsymbol{H}_{1}\right|=f_{x x}>0$ and $\left|\boldsymbol{H}_{2}\right|=f_{x x} f_{y y}>\left(f_{x y}\right)^{2}$.

### 11.3. OBJECTIVE FUNCTIONS WITH MORE THAN TWO VARIABLES179

Example 11.2.6 Find the extreme values of $z=8 x^{3}+2 x y-3 x^{2}+y^{2}+1$.

$$
\begin{gathered}
f_{x}=24 x^{2}+2 y-6 x, \quad f_{y}=2 x+2 y \\
f_{x x}=48 x-6, \quad f_{y y}=2, \quad f_{x y}=2 .
\end{gathered}
$$

Setting $f_{x}=0$ and $f_{y}=0$, we have

$$
\begin{gathered}
24 x^{2}+2 y-6 x=0 \\
2 y+2 x=0
\end{gathered}
$$

Then $y=-x$ and thus from $24 x^{2}+2 y-6 y$, we have $24 x^{2}-8 x=0$ which yields two solutions for $x$ : $\bar{x}_{1}=0$ and $\bar{x}_{2}=1 / 3$.

Since $f_{x x}(0,0)=-6$ and $f_{y y}(0,0)=2, f_{x x} f_{y y}-\left(f_{x y}\right)^{2}=-12-4=$ $-16<0$, so the point $\left(\bar{x}_{1}, \bar{y}_{1}\right)=(0,0)$ is not extreme point. For the solution $\left(\bar{x}_{2}, \bar{y}_{2}\right)=(1 / 3,-1 / 3)$, we find that $f_{x x}=10>0, f_{y y}=f_{x y}=2>0$, and $f_{x x} f_{y y}-\left(f_{x y}\right)^{2}=20-4>0$, so $(\bar{x}, \bar{y}, \bar{z})=(1 / 3,-1 / 3,23 / 27)$ is a relative minimum point.

Example 11.2.7 $z=x+2 e y-e^{x}-e^{2 y}$. Letting $f_{x}=1-e^{x}=0$ and $f_{y}=$ $2 e-2 e^{2 y}=0$, we have $\bar{x}=0$ and $\bar{y}=1 / 2$. Since $f_{x x}=-e^{x}, f_{y y}=-4 e^{2 y}$, and $f_{x y}=0$, then $f_{x x}(0,1 / 2)=-1<0$ and $f_{x x} f_{y y}-\left(f_{x y}\right)^{2}>0$. Therefore, $(\bar{x}, \bar{y}, \bar{z})=(0,1 / 2,-1)$ is the maximization of the function.

### 11.3 Objective Functions with More than Two Variables

When there are $n$ choice variables, the objective function may be expressed as

$$
z=f\left(x_{1}, x_{2}, \cdots, x_{n}\right)
$$

The total differential then is

$$
d z=f_{1} d x_{1}+f_{2} d x_{2}+\cdots+f_{n} d x_{n}
$$

so the necessary condition for an extremum is $d z=0$ for arbitrary $d x_{i}$. This, in turn, means that all $n$ first-order partial derivatives must be zero:

$$
f_{1}=f_{2}=\cdots=f_{n}=0
$$

It can be verified that the second-order differential $d^{2} z$ can be written as

$$
\begin{aligned}
d^{2} z & =\left[d x_{1}, d x_{2}, \cdots, d x_{n}\right]\left[\begin{array}{cccc}
f_{11} & f_{12} & \cdots & f_{1 n} \\
f_{21} & f_{22} & \cdots & f_{2 n} \\
\cdots & \cdots & \cdots & \cdots \\
f_{n 1} & f_{n 2} & \cdots & f_{n n}
\end{array}\right]\left[\begin{array}{c}
d x_{1} \\
d x_{2} \\
\cdots \\
d x_{n}
\end{array}\right] \\
& \equiv(d \boldsymbol{x})^{\prime} \boldsymbol{H} d \boldsymbol{x} .
\end{aligned}
$$

Thus the Hessian determinant is

$$
|\boldsymbol{H}|=\left|\begin{array}{cccc}
f_{11} & f_{12} & \cdots & f_{1 n} \\
f_{21} & f_{22} & \cdots & f_{2 n} \\
\cdots & \cdots & \cdots & \cdots \\
f_{n 1} & f_{n 2} & \cdots & f_{n n}
\end{array}\right|
$$

and the second-order sufficient condition for extremum is, as before, that

### 11.3. OBJECTIVE FUNCTIONS WITH MORE THAN TWO VARIABLES181

all the $n$ order of the leading principal minors, $k=1,2, \ldots, n$,

$$
|\boldsymbol{H}|=\left|\begin{array}{cccc}
f_{11} & f_{12} & \cdots & f_{1 k} \\
f_{21} & f_{22} & \cdots & f_{2 k} \\
\cdots & \cdots & \cdots & \cdots \\
f_{k 1} & f_{k 2} & \cdots & f_{k k}
\end{array}\right|
$$

is positive for a minimum in $z$ and that they duly alternate in sign for a maximum in $z$, the first one being negative.

In summary, we have the following proposition.

Proposition 11.3.1 (Conditions for Extremum) Suppose that $z=f\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ are twice continuously differentiable. Then, we have:

## Conditions for Maximum:

(1) $f_{1}=f_{2}=\cdots=f_{n}=0$ (necessary condition);
(2) $\left|\boldsymbol{H}_{1}\right|<0,\left|\boldsymbol{H}_{2}\right|>0,\left|\boldsymbol{H}_{3}\right|<0, \cdots,(-1)^{n}\left|\boldsymbol{H}_{n}\right|>0$. (i.e., $d^{2} z$ is negative definite.

## Conditions for Minimum:

(1) $f_{1}=f_{2}=\cdots=f_{n}=0$ (necessary condition);
(2) $\left|\boldsymbol{H}_{1}\right|>0,\left|\boldsymbol{H}_{2}\right|>0,\left|\boldsymbol{H}_{3}\right|>0, \cdots,\left|\boldsymbol{H}_{n}\right|>0$. (i.e., $d^{2} z$ is positive definite).

Example 11.3.1 Find the extreme values of

$$
z=2 x_{1}^{2}+x_{1} x_{2}+4 x_{2}^{2}+x_{1} x_{3}+x_{3}^{2}+2 .
$$

From the first-order condition:

$$
\begin{aligned}
& f_{1}=0: 4 x_{1}+x_{2}+x_{3}=0 \\
& f_{2}=0: x_{1}+8 x_{2}+0=0 \\
& f_{3}=0: x_{1}+0+2 x_{3}=0
\end{aligned}
$$

we find a unique solution $\bar{x}_{1}=\bar{x}_{2}=\bar{x}_{3}=0$. The Hessian determinant of this function is

$$
|\boldsymbol{H}|=\left|\begin{array}{lll}
f_{11} & f_{12} & f_{13} \\
f_{21} & f_{22} & f_{23} \\
f_{31} & f_{32} & f_{33}
\end{array}\right|=\left|\begin{array}{lll}
4 & 1 & 1 \\
1 & 8 & 0 \\
1 & 0 & 2
\end{array}\right| .
$$

Since the order of the leading principal minors of which are all positive: $\left|\boldsymbol{H}_{1}\right|=4>0,\left|\boldsymbol{H}_{2}\right|=31>0$, and $\left|\boldsymbol{H}_{3}\right|=54>0$, we can conclude that $\bar{z}=2$ is a minimum.

## Example 11.3.2 (Least Squares Estimator of Multiple Regression Model)

Consider the multiple regression model:

$$
\boldsymbol{y}=\boldsymbol{X} \boldsymbol{\beta}+\boldsymbol{\epsilon}
$$

where $\boldsymbol{y}$ is an $n \times 1$ vector of dependent variables, $\boldsymbol{X}$ is an $n \times k$ matrix of $k$ explanatory variables with $\operatorname{rank}(\boldsymbol{X})=k, \boldsymbol{\beta}$ is a $k \times 1$ vector of coefficients to be estimated, and $\boldsymbol{\epsilon}$ is an $n \times 1$ vector of disturbances. We assume that the matrices of observations $\boldsymbol{X}$ and $\boldsymbol{y}$ are given. Our goal is to find an estimator $\boldsymbol{b}$ for $\boldsymbol{\beta}$ using the least squares method.

The least squares estimator of $\beta$ is a vector $b$, which minimizes the sum of squared residuals, defined as:

$$
E(\boldsymbol{b})=(\boldsymbol{y}-\boldsymbol{X} \boldsymbol{b})^{\prime}(\boldsymbol{y}-\boldsymbol{X} \boldsymbol{b})=\boldsymbol{y}^{\prime} \boldsymbol{y}-\boldsymbol{y}^{\prime} \boldsymbol{X} \boldsymbol{b}-\boldsymbol{b}^{\prime} \boldsymbol{X}^{\prime} \boldsymbol{y}+\boldsymbol{b}^{\prime} \boldsymbol{X}^{\prime} \boldsymbol{X} \boldsymbol{b}
$$

By the formula for matrix differentiation in Chapter 8.7: $\frac{d a^{\prime} b}{d b}=a, \frac{d b^{\prime} a}{d b}=$
$\boldsymbol{a}$, and $\frac{d b^{\prime} \boldsymbol{M b}}{d \boldsymbol{b}}=\left(\boldsymbol{M}+\boldsymbol{M}^{\prime}\right) \boldsymbol{b}$, we know

$$
\begin{gathered}
\frac{d\left(\boldsymbol{y}^{\prime} \boldsymbol{X}\right) \boldsymbol{b}}{d \boldsymbol{b}}=\boldsymbol{X}^{\prime} \boldsymbol{y} \\
\frac{d \boldsymbol{b}^{\prime}\left(\boldsymbol{X}^{\prime} \boldsymbol{y}\right)}{d \boldsymbol{b}}=\boldsymbol{X}^{\prime} \boldsymbol{y} \\
\frac{d \boldsymbol{b}^{\prime} \boldsymbol{X}^{\prime} \boldsymbol{X} \boldsymbol{b}}{d \boldsymbol{b}}=2 \boldsymbol{X}^{\prime} \boldsymbol{X} .
\end{gathered}
$$

Therefore, the first-order condition is given by:

$$
\begin{equation*}
\frac{d E(\boldsymbol{b})}{d \boldsymbol{b}}=-2 \boldsymbol{X}^{\prime} \boldsymbol{y}+2 \boldsymbol{X}^{\prime} \boldsymbol{X} \boldsymbol{b}=\mathbf{0} \tag{11.3.1}
\end{equation*}
$$

and thus we have the estimator of $\beta$ :

$$
\boldsymbol{b}=\left(\boldsymbol{X}^{\prime} \boldsymbol{X}\right)^{-1} \boldsymbol{X}^{\prime} \boldsymbol{y}
$$

Differentiating (11.3.1) and using the formula for matrix differentiation again, we can obtain a matrix consisting of second-order partial derivatives:

$$
\frac{d^{2} E(\boldsymbol{b})}{d \boldsymbol{b}^{2}}=\left(2 \boldsymbol{X}^{\prime} \boldsymbol{X}\right)^{\prime}=2 \boldsymbol{X}^{\prime} \boldsymbol{X}
$$

To check whether the solution $\boldsymbol{b}$ is indeed a minimum, we need to prove the positive definiteness of the matrix $\boldsymbol{X}^{\prime} \boldsymbol{X}$. First, notice that $\boldsymbol{X}^{\prime} \boldsymbol{X}$ is a symmetric matrix. To prove positive definiteness, we take an arbitrary $k \times 1$ vector $\boldsymbol{z}, \boldsymbol{z} \neq \mathbf{0}$ and check the following quadratic form:

$$
\boldsymbol{z}^{\prime}\left(\boldsymbol{X}^{\prime} \boldsymbol{X}\right) \boldsymbol{z}=(\boldsymbol{X} \boldsymbol{z})^{\prime}(\boldsymbol{X} \boldsymbol{z})
$$

Since $\operatorname{rank}(\boldsymbol{X})=k$ and $\boldsymbol{z} \neq \mathbf{0}, \boldsymbol{X} \boldsymbol{z} \neq \mathbf{0}$. Thus, $\boldsymbol{X}^{\prime} X$ is positive definite.
Therefore, the square of the disturbance reaches its minimum:

$$
E(\overline{\boldsymbol{b}})=\boldsymbol{y}^{\prime}\left(\boldsymbol{I}-\boldsymbol{X}\left(\boldsymbol{X}^{\prime} \boldsymbol{X}\right)^{-1} \boldsymbol{X}^{\prime}\right) \boldsymbol{y}
$$

by noting that $\boldsymbol{A} \equiv\left(\boldsymbol{I}-\boldsymbol{X}\left(\boldsymbol{X}^{\prime} \boldsymbol{X}\right)^{-1} \boldsymbol{X}^{\prime}\right)$ is idempotent, i.e., $\boldsymbol{A} \boldsymbol{A}=\boldsymbol{A}$.

### 11.4 Second-Order Conditions in Relation to Concavity and Convexity

Second-order conditions are used to determine whether a stationary point of a function is a local maximum or minimum. These conditions are closely related to the concept of (strictly) concave and convex functions. A function is said to be concave (resp. convex) if its graph is shaped like a hill (resp. valley). However, a hill or valley may not exist over the entire domain, and we may only have a local maximum or minimum. In this section, we explore under what conditions a local maximum/minimum becomes a global maximum/minimum for a concave/convex function.

### 11.4.1 Concave and Convex Functions

A function that gives rise to a hill (resp. valley) over the entire domain is said to be a concave (resp. convex) function. If the hill (resp. valley) exists only on a subset $S$ of the domain, the function is said to be concave (resp. convex) on $S$. Formally, we have the following definition.

Definition 11.4.1 Suppose $X$ is a convex set. A function $f: X \rightarrow \mathbb{R}$ is said to be concave if, for any pair of distinct points $\boldsymbol{u}$ and $\boldsymbol{v}$ in $X$, and for any $0<\theta<1$,

$$
\theta f(\boldsymbol{u})+(1-\theta) f(\boldsymbol{v}) \leq f(\theta \boldsymbol{u}+(1-\theta) \boldsymbol{v})
$$

It is said to be convex if, for any pair of distinct points $\boldsymbol{u}$ and $\boldsymbol{v}$ in $X$, and for any $0<\theta<1$,

$$
\theta f(\boldsymbol{u})+(1-\theta) f(\boldsymbol{v}) \geq f(\theta \boldsymbol{u}+(1-\theta) \boldsymbol{v}) .
$$

Furthermore, if the weak inequality " $\leq$ " (resp. " $\geq$ ") is replaced by the strictly inequality " $<$ " (resp. " $>$ "), the function is said to be strictly concave (resp. strictly convex).

Remark 11.4.1 $\theta \boldsymbol{u}+(1-\theta) \boldsymbol{v}$ consists of line segments between points $\boldsymbol{u}$ and $\boldsymbol{v}$ when $0 \leq \theta \leq 1$. Thus, in the sense of geometry, the function $f$ is concave (resp. convex) if and only if the line segment of any two points $u$ and $\boldsymbol{v}$ lies on or below (resp. above) the surface. The function is strictly concave (resp. strictly convex) if and only if the line segment lies entirely below (resp. above) the surface, except at $M$ and $N$.


Figure 11.2: The graphical illustration of a concave function with two choice variables and the definition of concavity.

From the definition of concavity and convexity, we have the following three theorems:

Theorem I (Linear functions). If $f(\boldsymbol{x})$ is a linear function, then it is a concave function as well as a convex function, but not strictly so.

Theorem II (Negative of a function). If $f(\boldsymbol{x})$ is a (strictly) concave function, then $-f(\boldsymbol{x})$ is a (strictly) convex function, and vice versa.

Theorem III (Sum of functions). If $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$ are both concave (resp. convex) functions, then $f(\boldsymbol{x})+g(\boldsymbol{x})$ is a concave (resp. convex) function. Furthermore, if either one or both of them are strictly concave (resp. strictly convex), then $f(\boldsymbol{x})+g(\boldsymbol{x})$ is strictly concave (resp. convex).

In view of the association of concavity (resp. convexity) with a global hill (valley) configuration, an extremum of a concave (resp. convex) function must be a peak - a maximum (resp. a bottom - a minimum). Moreover, the maximum (resp. minimum) must be an absolute maximum (resp. minimum). Furthermore, the maximum (resp. minimum) is unique if the function is strictly concave (resp. strictly convex).

In the preceding paragraph, the properties of concavity and convexity are taken to be global in scope. If they are valid only for a portion of the surface (only in a subset $S$ of the domain $X$ ), the associated maximum and minimum are relative to that subset of the domain.

We know that when $z=f\left(x_{1}, \cdots, x_{n}\right)$ is twice continuously differentiable, $z=f\left(x_{1}, \cdots, x_{n}\right)$ reaches its maximum (resp. minimum) if $d^{2} z$ is negative (resp. positive) definite.

The following proposition shows the relationship between concavity (resp. convexity) and negative definiteness.

Proposition 11.4.1 Suppose that a function $z=f(\cdot): X \rightarrow \mathbb{R}$ is twice continuously differentiable on $X$. Then,
(1) the said function is concave (resp. convex) if and only if $d^{2} z$ is everywhere negative (resp. positive) semidefinite, or if and only if the eigenvalues of the Hessian matrix are all nonpositive (resp. nonnegative).
(2) it is strictly concave (resp. convex) if (but not only if) $d^{2} z$ is everywhere negative (resp. positive) definite, i.e., its Hessian matrix $\boldsymbol{H}=D^{2} f(\boldsymbol{x})$ is negative (positive) definite on $X$.

Remark 11.4.2 As discussed above, the strict concavity of a function $f(\boldsymbol{x})$ can be determined by testing whether the order of the leading principal minors of the Hessian matrix change signs alternately, namely,

$$
\begin{aligned}
& \left|\boldsymbol{H}_{1}\right|=f_{11}<0, \\
& \left|\boldsymbol{H}_{2}\right|=\left|\begin{array}{lll}
f_{11} & f_{12} \\
f_{21} & f_{22}
\end{array}\right|>0, \\
& \left|\boldsymbol{H}_{3}\right|=\left|\begin{array}{lll}
f_{11} & f_{12} & f_{13} \\
f_{21} & f_{22} & f_{23} \\
f_{31} & f_{32} & f_{33}
\end{array}\right|<0, \\
& \cdots, \\
& (-1)^{n}\left|\boldsymbol{H}_{n}\right|=(-1)^{n}|H|>0
\end{aligned}
$$

and so on, where $f_{i j}=\frac{\partial^{2} f}{\partial x_{i} \partial x_{j}}$. This algebraic condition is very useful for testing second-order conditions of optimality. It can easily verify whether a function is strictly concave (resp. strictly convex) by checking whether its Hessian matrix is negative (resp. positive) definite.

Example 11.4.1 Check $z=-x^{4}$ for concavity or convexity by the derivative condition.

Since $d^{2} z=-12 x^{2} d x^{2} \leq 0$ for all $x$ and $d x^{2}$, it is concave. This function, in fact, is strictly concave.

Example 11.4.2 Check $z=x_{1}^{2}+x_{2}^{2}$ for concavity or convexity.
Since

$$
|\boldsymbol{H}|=\left|\begin{array}{ll}
f_{11} & f_{12} \\
f_{21} & f_{22}
\end{array}\right|=\left|\begin{array}{ll}
2 & 0 \\
0 & 2
\end{array}\right|,
$$

$\left|\boldsymbol{H}_{1}\right|=2>0,\left|\boldsymbol{H}_{2}\right|=4>0$. Thus, by the proposition, the function is strictly convex.

Example 11.4.3 Check if the following production is concave:

$$
Q=f(L, K)=L^{\alpha} K^{\beta}
$$

where $L, K>0 ; \alpha, \beta>0$, and $\alpha+\beta<1$.
Since

$$
\begin{aligned}
f_{L} & =\alpha L^{\alpha-1} K^{\beta}, \\
f_{K} & =\beta L^{\alpha} L^{\beta-1} ; \\
f_{L L} & =\alpha(\alpha-1) L^{\alpha-2} K^{\beta}, \\
f_{K K} & =\beta(\beta-1) L^{\alpha} K^{\beta-2}, \\
f_{L K} & =\alpha \beta L^{\alpha-1} K^{\beta-1},
\end{aligned}
$$

thus

$$
\begin{aligned}
\left|\boldsymbol{H}_{1}\right| & =f_{L L}=\alpha(\alpha-1) L^{\alpha-2} K^{\beta}<0 ; \\
\left|\boldsymbol{H}_{2}\right| & =\left|\begin{array}{ll}
f_{L L} & f_{L K} \\
f_{K L} & f_{K K}
\end{array}\right|=f_{L L} f_{K K}-\left(f_{L K}\right)^{2} \\
& =\alpha \beta(\alpha-1)(\beta-1) L^{2(\alpha-1)} K^{2(\beta-1)}-\alpha^{2} \beta^{2} L^{2(\alpha-1)} K^{2(\beta-1)} \\
& =\alpha \beta[(\alpha-1)(\beta-1)-\alpha \beta] L^{2(\alpha-1)} K^{2(\beta-1)} \\
& =\alpha \beta(1-\alpha-\beta) L^{2(\alpha-1)} K^{2(\beta-1)}>0 .
\end{aligned}
$$

Therefore, it is strictly concave for $L, K>0, \alpha, \beta>0$, and $\alpha+\beta<1$.

If we only require a function to be differentiable, but not twice differentiable, the following proposition fully characterizes the concavity of the function:

Proposition 11.4.2 Let $X \subseteq \mathbb{R}$. Suppose that $f: X \rightarrow \mathbb{R}$ is differentiable. Then
$f$ is concave if and only if for any $x, y \in \mathbb{R}$, we have

$$
\begin{equation*}
f(y) \leq f(x)+f^{\prime}(x)(y-x) \tag{11.4.2}
\end{equation*}
$$

Indeed, for a concave function $u$, we can see from Figure 11.3 that

$$
\frac{u(x)-u\left(x^{*}\right)}{x-x^{*}} \leq u^{\prime}\left(x^{*}\right)
$$

which means (12.4.9).


Figure 11.3: The graphical illustration why Proposition 11.4.2 holds for a concave function.

When there are two or more independent variables, the above proposition becomes:

Proposition 11.4.3 Let $X \subseteq \mathbb{R}^{n}$. Suppose that $f: X \rightarrow \mathbb{R}$ is differentiable on $X$. Then $f$ is concave if and only if for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}$, we have

$$
\begin{equation*}
f(\boldsymbol{y}) \leq f(\boldsymbol{x})+\sum_{j=1}^{n} \frac{\partial f(\boldsymbol{x})}{\partial x_{j}}\left(y_{j}-x_{j}\right) . \tag{11.4.3}
\end{equation*}
$$

### 11.4.2 Concavity/Convexity and Global Optimization

In general, the local optimum is not necessarily the same as the global optimum. However, for a certain class of functions, these two are consistent with each other.

Theorem 11.4.1 (Local and Global Optimum) Suppose that $f$ is differentiable and concave on $X \subseteq \mathbb{R}^{n}$, and $\boldsymbol{x}^{*}$ is an interior point of $X$. Then, the following three statements are equivalent:
(1) $D f\left(\boldsymbol{x}^{*}\right)=0$.
(2) $f$ has a local maximum at $\boldsymbol{x}^{*}$.
(3) f has a global maximum at $\boldsymbol{x}^{*}$.

Proof. It is clear that $(3) \Rightarrow(2)$. Also, by the first-order condition, $(2) \Rightarrow(1)$. We just need to prove that $(1) \Rightarrow(3)$.

Suppose that $D f\left(\boldsymbol{x}^{*}\right)=0$. Then, the fact that $f$ is concave implies that for all $\boldsymbol{x}$ in the domain, we have:

$$
f(\boldsymbol{x}) \leq f\left(\boldsymbol{x}^{*}\right)+D f\left(\boldsymbol{x}^{*}\right)\left(\boldsymbol{x}-\boldsymbol{x}^{*}\right) .
$$

These two formulas mean that for all $\boldsymbol{x}$, we must have

$$
f(\boldsymbol{x}) \leq f\left(\boldsymbol{x}^{*}\right)
$$

Therefore, $f$ reaches a global maximum at $\boldsymbol{x}^{*}$.
When the function is strictly concave (strictly convex), we have the uniqueness of the optimum. Formally, we have the following theorem.

Theorem 11.4.2 (Uniqueness of Global Optimum) Let $X \subseteq \mathbb{R}^{n}$.
(1) If a strictly concave function $f$ defined on $X$ reaches a local maximum at $\boldsymbol{x}^{*}$, then $f\left(\boldsymbol{x}^{*}\right)$ is the unique global maximum.
(2) If a strictly convex function $f$ reaches a local minimum at $\tilde{\boldsymbol{x}}$, then $f(\tilde{\boldsymbol{x}})$ is the unique global minimuт.

Proof. Proof by contradiction. If $f$ reaches a local maximum at $\boldsymbol{x}^{*}$ but not unique, then there is a point $\boldsymbol{x}^{\prime} \neq \boldsymbol{x}^{*}$ such that $f\left(\boldsymbol{x}^{\prime}\right)=f\left(\boldsymbol{x}^{*}\right)$. Suppose that $\boldsymbol{x}^{t}=t \boldsymbol{x}^{\prime}+(1-t) \boldsymbol{x}^{*}$. Then, strict concavity requires that for all $t \in(0,1)$,

$$
f\left(\boldsymbol{x}^{t}\right)>t f\left(\boldsymbol{x}^{\prime}\right)+(1-t) f\left(\boldsymbol{x}^{*}\right) .
$$

Since $f\left(\boldsymbol{x}^{\prime}\right)=f\left(\boldsymbol{x}^{*}\right)$,

$$
f\left(\boldsymbol{x}^{t}\right)>t f\left(\boldsymbol{x}^{\prime}\right)+(1-t) f\left(\boldsymbol{x}^{\prime}\right)=f\left(\boldsymbol{x}^{\prime}\right) .
$$

This contradicts the assumption that $f\left(\boldsymbol{x}^{\prime}\right)$ is a global maximum. Consequently, the global maximum of a strictly concave function is unique. The proof of part (2) is similar and thus omitted.

### 11.5 Economic Applications

## Problem of a Multiproduct Firm

Example 11.5.1 Suppose that a competitive firm produces two products. Let $Q_{i}$ represent the output level of the $i$-th product, and let the prices of the products be denoted by $P_{1}$ and $P_{2}$. Since the firm is a competitive firm, it takes the prices as given. Then, the firm's revenue function is

$$
T R=P_{1} Q_{1}+P_{2} Q_{2}
$$

The firm's cost function is assumed to be

$$
C=2 Q_{1}^{2}+Q_{1} Q_{2}+2 Q_{2}^{2}
$$

Then, the profit function of this hypothetical firm is given by

$$
\pi=T R-C=P_{1} Q_{1}+P_{2} Q_{2}-2 Q_{1}^{2}-Q_{1} Q_{2}-2 Q_{2}^{2}
$$

The firm wants to maximize the profit by choosing the levels of $Q_{1}$ and $Q_{2}$. For this purpose, setting

$$
\begin{aligned}
& \frac{\partial \pi}{\partial Q_{1}}=0: P_{1}-4 Q_{1}-Q_{2}=0 \\
& \frac{\partial \pi}{\partial Q_{2}}=0: P_{2}-Q_{1}-4 Q_{2}=0
\end{aligned}
$$

we have

$$
\begin{aligned}
& 4 Q_{1}+Q_{2}=P_{1} \\
& Q_{1}+4 Q_{2}=P_{2}
\end{aligned}
$$

and thus

$$
\bar{Q}_{1}=\frac{4 P_{1}-P_{2}}{15} \text { and } \bar{Q}_{2}=\frac{4 P_{2}-P_{1}}{15} .
$$

Also, the Hessian matrix is

$$
\boldsymbol{H}=\left[\begin{array}{ll}
\pi_{11} & \pi_{12} \\
\pi_{21} & \pi_{22}
\end{array}\right]=\left[\begin{array}{ll}
-4 & -1 \\
-1 & -4
\end{array}\right] .
$$

Since $\left|\boldsymbol{H}_{1}\right|=-4<0$ and $\left|\boldsymbol{H}_{2}\right|=16-1>0$, the Hessian matrix (or $d^{2} z$ ) is negative definite, and the solution does maximize. In fact, since $H$ is everywhere negative definite, the maximum profit found above is actually a unique absolute maximum.

Example 11.5.2 Let us now transplant the problem in the above example into the setting of a monopolistic market.

Suppose that the demands facing the monopolist firm are as follows:

$$
\begin{aligned}
& Q_{1}=40-2 P_{1}+P_{2} \\
& Q_{2}=15+P_{1}-P_{2}
\end{aligned}
$$

Again, the cost function is given by

$$
C=Q_{1}^{2}+Q_{1} Q_{2}+Q_{2}^{2}
$$

From the monopolist's demand function, we can express prices $P_{1}$ and $P_{2}$ as functions of $Q_{1}$ and $Q_{2}$, and so for the profit function. The reason we want to do so is that we need to express the profit as the function of outputs only. Thus, solving

$$
\begin{aligned}
& -2 P_{1}+P_{2}=Q_{1}-40 \\
& P_{1}-P_{2}=Q_{2}-15,
\end{aligned}
$$

we have

$$
\begin{aligned}
& P_{1}=55-Q_{1}-Q_{2} ; \\
& P_{2}=70-Q_{1}-2 Q_{2} .
\end{aligned}
$$

Consequently, the firm's total revenue function $T R$ can be written as

$$
\begin{aligned}
T R & =P_{1} Q_{1}+P_{2} Q_{2} \\
& =\left(55-Q_{1}-Q_{2}\right) Q_{1}+\left(70-Q_{1}-2 Q_{2}\right) Q_{2} ; \\
& =55 Q_{1}+70 Q_{2}-2 Q_{1} Q_{2}-Q_{1}^{2}-2 Q_{2}^{2} .
\end{aligned}
$$

Thus, the profit function is

$$
\begin{aligned}
\pi & =T R-C \\
& =55 Q_{1}+70 Q_{2}-3 Q_{1} Q_{2}-2 Q_{1}^{2}-3 Q_{2}^{2}
\end{aligned}
$$

which is an object function with two choice variables $Q_{1}$ and $Q_{2}$. Setting

$$
\begin{aligned}
& \frac{\partial \pi}{\partial Q_{1}}=0: 55-4 Q_{1}-3 Q_{2}=0 \\
& \frac{\partial \pi}{\partial Q_{2}}=0: 70-3 Q_{1}-6 Q_{2}=0
\end{aligned}
$$

we can find the solution output level is

$$
\left(\bar{Q}_{1}, \bar{Q}_{2}\right)=\left(8,7 \frac{2}{3}\right)
$$

The prices and profit are

$$
\bar{P}_{1}=39 \frac{1}{3}, \bar{P}_{2}=46 \frac{2}{3}, \text { and } \bar{\pi}=488 \frac{1}{3} .
$$

Inasmuch as the Hessian determinant is

$$
\left|\begin{array}{ll}
-4 & -3 \\
-3 & -6
\end{array}\right|
$$

thus $\left|H_{1}\right|=-4<0$ and $\left|H_{2}\right|=15>0$. So, Hessian matrix is everywhere negative definite, and $\bar{\pi}$ does represent a unique absolute maximum.

## Chapter 12

## Optimization with Equality

## Constraints

The previous chapter presented a general method for finding the relative extrema of an objective function of two or more variables. One important feature of that discussion is that all the choice variables are independent of one another, in the sense that the decision made regarding one variable does not depend on the choice of the remaining variables. However, in many optimization problems, variables are subject to constraints. For instance, a consumer maximizes her utility subject to her budget constraint, and a firm minimizes the cost of production with the constraint of production technique.

In this chapter, we shall consider the problem of optimization with equality constraints. Our primary concern will be with finding the relative constrained extrema.

### 12.1 Effects of a Constraint

In general, for a function $z=f(x, y)$, the difference between a constrained extremum and a free extremum can be illustrated in Figure 12.1.


Figure 12.1: Difference between a constrained extremum and a free extremum

The free extremum in this particular graph is the peak point of the entire domain, but the constrained extremum is at the peak of the inverse U shaped curve situated on top of the constraint line. In general, a constraint (less freedom) maximum can be expected to have a lower value than the free (more freedom) maximum, although by coincidence, the two maxima may happen to have the same value. But the constrained maximum can never exceed the free maximum. To have a certain degree of freedom of choices, the number of constraints should be less than the number of choice variables.

### 12.2 Finding the Stationary Values

For the purpose of illustration, let us consider a consumer choice problem: maximizing the utility function:

$$
u\left(x_{1}, x_{2}\right)=x_{1} x_{2}+2 x_{1}
$$

subject to the budget constraint:

$$
4 x_{1}+2 x_{2}=60 .
$$

Even without any new technique of solution, the constrained maximum in this problem can easily be found. Since the budget line implies

$$
x_{2}=\frac{60-4 x_{1}}{2}=30-2 x_{1},
$$

we can combine the constraint with the objective function by substitution. The result is an objective function in one variable only:

$$
u=x_{1}\left(30-2 x_{1}\right)+2 x_{1}=32 x_{1}-2 x_{1}^{2},
$$

which can be handled with the method already learned. By setting

$$
\frac{\partial u}{\partial x_{1}}=32-4 x_{1}=0
$$

we obtain the solution $\bar{x}_{1}=8$, and thus, by the budget constraint, $\bar{x}_{2}=$ $30-2 \bar{x}_{1}=30-16=14$. Since $\frac{d^{2} u}{d x_{1}^{2}}=-4<0$, this stationary value constitutes a (constrained) maximum.

However, when the constraint itself is a complicated function, or when the constraint cannot be solved to express one variable as an explicit function of the other variables, the technique of substitution and elimination
of variables could become a burdensome task or would, in fact, be of no avail. In such cases, we may resort to a method known as the method of Lagrange multipliers.

## Lagrange-Multiplier Method

The essence of the Lagrange-multiplier method is to convert a constrained extremum problem into a free-extremum problem so that the firstorder condition approach can still be applied.

In general, given an objective function

$$
\begin{equation*}
z=f(x, y) \tag{12.2.1}
\end{equation*}
$$

subject to the constraint

$$
\begin{equation*}
g(x, y)=c, \tag{12.2.2}
\end{equation*}
$$

where $c$ is a constant, we can define the Lagrange function as

$$
Z=f(x, y)+\lambda[c-g(x, y)],
$$

where the symbol $\lambda$ represents an unknown constant called the Lagrange multiplier.

If we can somehow be assured that $g(x, y)=c$, the last term of $Z$ will vanish regardless of the value of $\lambda$. In that case, $Z$ will be identical to $u$, and with the constraint removed, we only need to find the free maximum. The question is: how can we make the parenthetical expression in $Z$ vanish?

The tactic that will accomplish this is to treat $\lambda$ as an additional variable, that is, to consider $Z=Z(\lambda, x, y)$. For stationary values of $Z$, the
first-order condition for an interior free extremum is:

$$
\begin{aligned}
Z_{\lambda} & \equiv \frac{\partial Z}{\partial \lambda}=c-g(x, y)=0 ; \\
Z_{x} & \equiv \frac{\partial Z}{\partial x}=f_{x}-\lambda g_{x}(x, y)=0 ; \\
Z_{y} & \equiv \frac{\partial Z}{\partial y}=f_{y}-\lambda g_{y}(x, y)=0 .
\end{aligned}
$$

Example 12.2.1 Let us again consider the consumer's choice problem above. The Lagrange function is

$$
Z=x_{1} x_{2}+2 x_{1}+\lambda\left[60-4 x_{1}-2 x_{2},\right]
$$

for which the necessary condition for a stationary value is

$$
\begin{aligned}
& Z_{\lambda}=60-4 x_{1}-2 x_{2}=0 \\
& Z_{x_{1}}=x_{2}+2-4 \lambda=0 \\
& Z_{x_{2}}=x_{1}-2 \lambda=0 .
\end{aligned}
$$

Solving the stationary point of the variables, we find that $\bar{x}_{1}=8, \bar{x}_{2}=14$, and $\lambda=4$. As expected, $\bar{x}_{1}=8$ and $\bar{x}_{2}=14$ are the same obtained by the substitution method.

Example 12.2.2 Find the extremum of $z=x y$ subject to $x+y=6$. The Lagrange function is

$$
Z=x y+\lambda(6-x-y) .
$$

The first-order condition is

$$
\begin{aligned}
& Z_{\lambda}=6-x-y=0 \\
& Z_{x}=y-\lambda=0 \\
& Z_{y}=x-\lambda=0 .
\end{aligned}
$$

Thus, we find $\bar{\lambda}=3, \bar{x}=3, \bar{y}=3$.

Example 12.2.3 Find the extremum of $z=x_{1}^{2}+x_{2}^{2}$ subject to $x_{1}+4 x_{2}=2$. The Lagrange function is

$$
Z=x_{1}^{2}+x_{2}^{2}+\lambda\left(2-x_{1}-4 x_{2}\right)
$$

The first-order condition (FOC) is

$$
\begin{aligned}
& Z_{\lambda}=2-x_{1}-4 x_{2}=0 ; \\
& Z_{x_{1}}=2 x_{1}-\lambda=0 ; \\
& Z_{x_{2}}=2 x_{2}-4 \lambda=0 .
\end{aligned}
$$

The stationary value of $Z$, defined by the solution

$$
\bar{\lambda}=\frac{4}{17}, \bar{x}_{1}=\frac{2}{17}, \bar{x}_{2}=\frac{8}{17},
$$

is therefore $\bar{Z}=\bar{z}=\frac{4}{17}$.
To tell whether $\bar{z}$ is a maximum, we need to consider the second-order condition.

## An Interpretation of the Lagrange Multiplier

The Lagrange multiplier $\bar{\lambda}$ measures the sensitivity of $Z$ to change in the constraint. If we can express the solution $\bar{\lambda}, \bar{x}$, and $\bar{y}$ all as implicit functions of the parameter $c$ :

$$
\bar{\lambda}=\bar{\lambda}(c), \bar{x}=\bar{x}(c), \text { and } \bar{y}=\bar{y}(c),
$$

all of which will have continuous derivative, we have the identities:

$$
\begin{aligned}
& c-g(\bar{x}, \bar{y}) \equiv 0 ; \\
& f_{x}(\bar{x}, \bar{y})-\bar{\lambda} g_{x}(\bar{x}, \bar{y}) \equiv 0 ; \\
& f_{y}(\bar{x}, \bar{y})-\bar{\lambda} g_{y}(\bar{x}, \bar{y}) \equiv 0 .
\end{aligned}
$$

Thus, we can consider $Z$ as a function of $c$ :

$$
Z=f(\bar{x}, \bar{y})+\bar{\lambda}[c-g(\bar{x}, \bar{y})] .
$$

Therefore, we have

$$
\begin{aligned}
\frac{d \bar{Z}}{d c} & =f_{x} \frac{d \bar{x}}{d c}+f_{y} \frac{d \bar{y}}{d c}+[c-g(\bar{x}, \bar{y})] \frac{d \bar{\lambda}}{d c}+\lambda\left[1-g_{x} \frac{d \bar{x}}{d c}-g_{y} \frac{d \bar{y}}{d c}\right] \\
& =\left(f_{x}-\lambda g_{x}\right) \frac{d \bar{x}}{d c}+\left(f_{y}-\lambda g_{y}\right) \frac{d \bar{y}}{d c}+[c-g(\bar{x}, \bar{y})] \frac{d \bar{x}}{d c}+\lambda \\
& =\lambda .
\end{aligned}
$$

## $n$-Variable and Multiconstraint Cases

The Lagrange-multiplier method can be easily generalized to cases with multiple constraints and $n$ variables.

$$
\begin{equation*}
z=f\left(x_{1}, x_{2}, \cdots, x_{n}\right) \tag{12.2.3}
\end{equation*}
$$

subject to

$$
\begin{equation*}
g\left(x_{1}, x_{2}, \cdots, x_{n}\right)=c . \tag{12.2.4}
\end{equation*}
$$

It follows that the Lagrange function is

$$
Z=f\left(x_{1}, x_{2}, \cdots, x_{n}\right)+\lambda\left[c-g\left(x_{1}, x_{2}, \cdots, x_{n}\right)\right],
$$ for which the first-order condition is given by

$$
\begin{aligned}
& Z_{\lambda}=c-g\left(x_{1}, x_{2}, \cdots, x_{n}\right)=0 \\
& Z_{i}=f_{i}\left(x_{1}, x_{2}, \cdots, x_{n}\right)-\lambda g_{i}\left(x_{1}, x_{2}, \cdots, x_{n}\right)=0 \quad[i=1,2, \cdots, n] .
\end{aligned}
$$

If the objective function has more than one constraint, say, two constraints

$$
g\left(x_{1}, x_{2}, \cdots, x_{n}\right)=c \text { and } h\left(x_{1}, x_{2}, \cdots, x_{n}\right)=d
$$

The Lagrange function is then defined by

$$
Z=f\left(x_{1}, x_{2}, \cdots, x_{n}\right)+\lambda\left[c-g\left(x_{1}, x_{2}, \cdots, x_{n}\right)\right]+\mu\left[d-h\left(x_{1}, x_{2}, \cdots, x_{n}\right)\right]
$$

for which the first-order condition consists of $(n+2)$ equations:

$$
\begin{aligned}
& Z_{\lambda}=c-g\left(x_{1}, x_{2}, \cdots, x_{n}\right)=0 \\
& Z_{\mu}=d-h\left(x_{1}, x_{2}, \cdots, x_{n}\right)=0 \\
& Z_{i}=f_{i}\left(x_{1}, x_{2}, \cdots, x_{n}\right)-\lambda g_{i}\left(x_{1}, x_{2}, \cdots, x_{n}\right)-\mu h_{i}\left(x_{1}, x_{2}, \cdots, x_{n}\right)=0 .
\end{aligned}
$$

Similarly, consider the most general setting of the problem with $n$ variables and $m$ constraints:

$$
\begin{aligned}
\text { extremize } & f\left(x_{1}, \ldots, x_{n}\right) \\
\text { s.t. } & g^{j}\left(x_{1}, \ldots, x_{n}\right)=b_{j}, j=1,2, \ldots, m<n .
\end{aligned}
$$

The difference $n-m$ is the number of degrees of freedom of the problem. Note that we must require that $n>m$, otherwise there is no degree of freedom to choose.

The Lagrangian function is given by

$$
\begin{equation*}
Z\left(x_{1}, \ldots, x_{n}, \lambda_{1}, \ldots, \lambda_{m}\right)=f\left(x_{1}, \ldots, x_{n}\right)+\sum_{j=1}^{m} \lambda_{j}\left(b_{j}-g^{j}\left(x_{1}, \ldots, x_{n}\right)\right) . \tag{12.2.6}
\end{equation*}
$$

Then we have the following conclusion regarding the equality constrained optimization problems.

## Proposition 12.2.1 (First-Order Necessary Condition for Interior Extremum)

Suppose that $f(\boldsymbol{x})$ and $g^{j}(\boldsymbol{x}), j=1, \cdots, m$, are continuously differentiable functions defined on $X \subseteq \mathbb{R}^{n}$ with $m<n, \boldsymbol{x}^{*}$ is an interior point of $X$ and an extreme point (maximal or minimal point) of $f$ ——where $f$ is subject to the constraint of $g^{j}\left(\boldsymbol{x}^{*}\right)=0$, where $j=1, \cdots, m$. If the gradient $D g^{j}\left(\boldsymbol{x}^{*}\right)=0, j=1, \cdots, m$, are linearly independent, then there is a unique $\lambda_{j}^{*}, j=1, \cdots, m$, such that:

$$
\frac{\partial Z\left(\boldsymbol{x}^{*}, \lambda^{*}\right)}{\partial x_{i}}=\frac{\partial f\left(\boldsymbol{x}^{*}\right)}{\partial x_{i}}+\sum_{i=1}^{m} \lambda_{j}^{*} \frac{\partial g^{j}\left(\boldsymbol{x}^{*}\right)}{\partial x_{i}}=0, \quad i=1, \cdots, n .
$$

### 12.3 Second-Order Conditions

From the previous section, we know that finding the constrained extremum is equivalent to finding the free extremum of the Lagrange function $Z$ and giving the first-order condition. This section gives the second-order sufficient condition for the constrained extremum of $f$.

For a constrained extremum of $z=f(x, y)$, subject to $g(x, y)=c$, the second-order necessary-and-sufficient conditions still revolve around the algebraic sign of the second-order total differential $d^{2} z$, evaluated at the stationary point. However, there is one important change. In the present context, we are concerned with the sign definiteness or semidefiniteness of $d^{2} z$, not for all possible values of $d x$ and $d y$ (not both zero), but only
for those $d x$ and $d y$ values (not both zero) satisfying the linear constraint $g_{x} d x+g_{y} d y=0$.

The second-order sufficient conditions are:

- For maximum of $z: d^{2} z$ is negative definite subject to $d g=0$.
- For minimum of $z: d^{2} z$ is positive definite subject to $d g=0$.

As the $(d x, d y)$ pairs satisfying the constraint $g_{x} d x+g_{y} d y=0$ constitute merely a subset of the set of all possible $d x$ and $d y$, the constrained sign definiteness is less stringent. In other words, the second-order sufficient condition for a constrained-extremum problem is a weaker condition than that for a free-extremum problem.

In the following, we shall concentrate on the second-order sufficient condition in terms of the bordered Hessian.

## The Bordered Hessian

In the case of constrained extrema, the second-order sufficient condition can be expressed in terms of the bordered Hessian instead of the Hessian determinant $|\boldsymbol{H}|$.

To develop this method, let us first analyze the conditions for the definiteness of a two-variable quadratic form

$$
q=d_{11} u^{2}+2 d_{12} u v+d_{22} v^{2}
$$

subject to a linear constraint

$$
a u+b v=0 .
$$

Substituting $v=-\frac{a}{b} u$ into the quadratic form, we have:

$$
\begin{align*}
q & =d_{11} u^{2}-2 d_{12} \frac{a}{b} u+d_{22} \frac{a^{2}}{b^{2}} u^{2} \\
& =\left[d_{11} b^{2}-2 d_{12} a b+d_{22} a^{2}\right] \frac{u^{2}}{b^{2}} \tag{12.3.7}
\end{align*}
$$

Then $q$ is positive (negative) definite subject to $a u+b v=0$ if and only if the expression in parentheses is positive (negative). However, it is worth noting that the following bordered determinant

$$
|\bar{H}| \equiv\left|\begin{array}{ccc}
0 & a & b \\
a & d_{11} & d_{12} \\
b & d_{21} & b_{22}
\end{array}\right|=-\left[d_{11} b^{2}-2 d_{12} a b+d_{22} a^{2}\right]
$$

which is exactly the negative of the term in parentheses. Consequently, $q$ is positive (negative) definite subject to $a u+b v=0$ if and only if $|\bar{H}|<0$ ( $|\bar{H}|>0)$.

Parallel to the constrained-extremum problem (12.2.1) of a function $z$ with two variables and constraint (12.2.2), the second-order sufficient condition for a maximum of $z$ reduces to

$$
|\bar{H}|=\left|\begin{array}{ccc}
0 & g_{x} & g_{y} \\
g_{x} & Z_{x x} & Z_{x y} \\
g_{y} & Z_{y x} & Z_{y y}
\end{array}\right|>0,
$$

and the second-order sufficient condition for minimum of $z$ reduces to

$$
|\bar{H}|=\left|\begin{array}{ccc}
0 & g_{x} & g_{y} \\
g_{x} & Z_{x x} & Z_{x y} \\
g_{y} & Z_{y x} & Z_{y y}
\end{array}\right|<0,
$$

where in the newly introduced symbols, the horizontal bar above $\boldsymbol{H}$ mean-
s bordered, and $Z_{i j}=f_{i j}-\lambda g_{i j}$ with $i, j=x, y$.
Now consider a general case where the objective functions take form

$$
z=f\left(x_{1}, x_{2}, \cdots, x_{n}\right)
$$

subject to

$$
g\left(x_{1}, x_{2}, \cdots, x_{n}\right)=c .
$$

The Lagrange function is then

$$
Z=f\left(x_{1}, x_{2}, \cdots, x_{n}\right)+\lambda\left[c-g\left(x_{1}, x_{2}, \cdots, x_{n}\right)\right] .
$$

Similarly, the bordered Hessian determinant $|\overline{\boldsymbol{H}}|$ is given by

$$
|\overline{\boldsymbol{H}}|=\left|\begin{array}{ccccc}
0 & g_{1} & g_{2} & \cdots & g_{n} \\
g_{1} & Z_{11} & Z_{12} & \cdots & Z_{1 n} \\
g_{2} & Z_{21} & Z_{22} & \cdots & Z_{2 n} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
g_{n} & Z_{n 1} & Z_{n 2} & \cdots & Z_{n n}
\end{array}\right|
$$

The order of the leading principal minors of the boarded Hessian can be defined as

$$
\left|\overline{\boldsymbol{H}}_{2}\right|=\left|\begin{array}{ccc}
0 & g_{1} & g_{2} \\
g_{1} & Z_{11} & Z_{12} \\
g_{2} & Z_{21} & Z_{22}
\end{array}\right|, \quad\left|\overline{\boldsymbol{H}}_{3}\right|=\left|\begin{array}{cccc}
0 & g_{1} & g_{2} & g_{3} \\
g_{1} & Z_{11} & Z_{12} & Z_{13} \\
g_{2} & Z_{21} & Z_{22} & Z_{23} \\
g_{3} & Z_{31} & Z_{32} & Z_{33}
\end{array}\right| \quad \text { (etc.) }
$$

with the last one being $\left|\overline{\boldsymbol{H}}_{n}\right|=|\overline{\boldsymbol{H}}|$, where the subscript indicates the order of the leading principal minor being bordered. For instance, $\left|\overline{\boldsymbol{H}}_{2}\right|$ involves the second leading principal minor of the (plain) Hessian, bordered with $0, g_{1}$, and $g_{2}$; and similarly for the others. The conditions for positive and
negative definiteness of $d^{2} z$ are then:
$d^{2} z$ is negative definite subject to $d g=\mathbf{0}$ iff

$$
\left|\overline{\boldsymbol{H}}_{2}\right|>0,\left|\overline{\boldsymbol{H}}_{3}\right|<0,\left|\overline{\boldsymbol{H}}_{4}\right|>0, \cdots,(-1)^{n}\left|\overline{\boldsymbol{H}}_{n}\right|>0,
$$

and $d^{2} z$ is positive definite subject to $d g=\mathbf{0}$ iff

$$
\left|\overline{\boldsymbol{H}}_{2}\right|<0,\left|\overline{\boldsymbol{H}}_{3}\right|<0,\left|\overline{\boldsymbol{H}}_{4}\right|<0, \cdots,\left(\left|\bar{H}_{n}\right|<0 .\right.
$$

As previously, a negative definite $d^{2} z$ is sufficient to establish a stationary value of $\mathbf{z}$ as its maximum, whereas a positive definite $d^{2} z$ is sufficient to establish it as a minimum.

Summarizing the above discussions, we have the following conclusions.

## Proposition 12.3.1 (Second-Order Sufficient Condition for Interior Extremum)

Suppose that $z=f\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ are twice continuously differentiable and $g\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ is differentiable. Then we have:

## The Conditions for Maximum:

(1) $Z_{\lambda}=Z_{1}=Z_{2}=\cdots=Z_{n}=0$ (necessary condition);
(2) $\left|\overline{\boldsymbol{H}}_{2}\right|>0,\left|\overline{\boldsymbol{H}}_{3}\right|<0,\left|\overline{\boldsymbol{H}}_{4}\right|>0, \cdots,(-1)^{n}\left|\bar{H}_{n}\right|>0$.

## The Conditions for Minimum:

(1) $Z_{\lambda}=Z_{1}=Z_{2}=\cdots=Z_{n}=0$ (necessary condition);
(2) $\left|\overline{\boldsymbol{H}}_{2}\right|<0,\left|\overline{\boldsymbol{H}}_{3}\right|<0,\left|\overline{\boldsymbol{H}}_{4}\right|<0, \cdots,\left(\left|\overline{\boldsymbol{H}}_{n}\right|<0\right.$.

Example 12.3.1 For the objective function $z=x y$ subject to $x+y=6$, we have shown that $(\bar{x}, \bar{y}, \bar{z})=(3,3,9)$ is a possible extremum solution. Since $Z_{x}=y-\lambda$ and $Z_{y}=x-\lambda$, then $Z_{x x}=0, Z_{x y}=1$, and $Z_{y y}=0, g_{x}=g_{y}=1$.

Thus, we find that

$$
|\overline{\boldsymbol{H}}|=\left|\begin{array}{lll}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{array}\right|=2>0
$$

which establishes the stationary value of $\bar{z}=9$ as a maximum.

Example 12.3.2 For the objective function $z=x_{1}^{2}+x_{2}^{2}$ subject to the constraint $x_{1}+4 x_{2}=2$, we have shown that $(\bar{x}, \bar{y}, \bar{z})=(2 / 17,8 / 17,4 / 17)$ is a possible extremum solution. To determine whether it is a maximum, minimum, or neither, we need to check the second-order sufficient condition.

Since $Z_{1}=2 x_{1}-\lambda$ and $Z_{2}=2 x_{2}-\lambda$ as well as $g_{1}=1$ and $g_{2}=4$, we have $Z_{11}=2, Z_{22}=2$, and $Z_{12}=Z_{21}=0$. It thus follows that the bordered Hessian is

$$
|\overline{\boldsymbol{H}}|=\left|\begin{array}{lll}
0 & 1 & 4 \\
1 & 2 & 0 \\
4 & 0 & 2
\end{array}\right|=-34<0
$$

which is negative. Thus, the extremum point $(\bar{x}, \bar{y}, \bar{z})=(2 / 17,8 / 17,4 / 17)$ is a local minimum of the objective function.

## Multiconstraint Case

When more than one constraint appears in the problem, the second-order condition involves a Hessian with more than one border.

Proposition 12.3.2 (Sufficient Condition with Multiple Constraints) Suppose that $f$ and $g^{1}, \ldots, g^{m}$ are twice continuously differentiable functions and $x^{*}$ satisfies the necessary condition for the problem (12.2.5). Define the bordered Hessian
$\left|\bar{H}_{r}\right|$ as

$$
\left|\overline{\boldsymbol{H}}_{r}\right|=\operatorname{det}\left(\begin{array}{cccccc}
0 & \cdots & 0 & \frac{\partial g^{1}}{\partial x_{1}} & \cdots & \frac{\partial g^{1}}{\partial x_{r}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \frac{\partial g^{m}}{\partial x_{1}} & \cdots & \frac{\partial g^{m}}{\partial x_{r}} \\
\frac{\partial g^{1}}{\partial x_{1}} & \cdots & \frac{\partial g^{m}}{\partial x_{1}} & \frac{\partial^{2} Z}{\partial x_{1} \partial x_{1}} & \cdots & \frac{\partial^{2} Z}{\partial x_{1} \partial x_{r}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial g^{1}}{\partial x_{r}} & \cdots & \frac{\partial g^{m}}{\partial x_{r}} & \frac{\partial^{2} Z}{\partial x_{r} \partial x_{1}} & \cdots & \frac{\partial^{2} Z}{\partial x_{r} \partial x_{r}}
\end{array}\right), \quad r=1,2, \ldots, n .
$$

Let $\left|\overline{\boldsymbol{H}}_{r}\left(x^{*}\right)\right|$ be the bordered Hessian determinant evaluated at $x^{*}$. Then
(1) if $(-1)^{r-m+1}\left|\overline{\boldsymbol{H}}_{r}\left(x^{*}\right)\right|>0, r=m+1, \ldots, n$, then $x^{*}$ is a local maximum point for the problem (12.2.5);
(2) if $(-1)^{m}\left|\overline{\boldsymbol{H}}_{r}\left(x^{*}\right)\right|>0, r=m+1, \ldots, n$, then $x^{*}$ is a local minimum point for the problem (12.2.5).

Note that when $m=1$, the above second-order sufficient condition reduces to the second-order sufficient condition with only one constraint.

### 12.4 Quasiconcavity and Quasiconvexity

For a problem of free extremum, we know that the concavity (resp. convexity) of the objective function guarantees the existence of absolute maximum (resp. absolute minimum). For a problem of constrained optimization, we will demonstrate that the quasiconcavity (resp. quasiconvexity) of the objective function guarantees the existence of a global maximum (resp. global minimum).

## Algebraic Characterization

Quasiconcavity and quasiconvexity, like concavity and convexity, can be either strict or non-strict:

Definition 12.4.1 A function is quasiconcave if, for any pair of distict points $\boldsymbol{u}$ and $\boldsymbol{v}$ in the convex domain of $f$, and for $0<\theta<1$, we have

$$
f(\theta \boldsymbol{u}+(1-\theta) \boldsymbol{v}) \geq \min \{f(\boldsymbol{u}), f(\boldsymbol{v})\}
$$

It is quasiconvex if, for any pair of distict points $\boldsymbol{u}$ and $\boldsymbol{v}$ in the convex domain of $f$, and for $0<\theta<1$, we have

$$
f(\theta \boldsymbol{u}+(1-\theta) \boldsymbol{v}) \leq \max \{f(\boldsymbol{u}), f(\boldsymbol{v})\} .
$$

Note that when $f(\boldsymbol{v}) \geq f(\boldsymbol{u})$, the above inequalities imply respectively

$$
\begin{array}{r}
f(\theta \boldsymbol{u}+(1-\theta) \boldsymbol{v}) \geq f(\boldsymbol{u}) \\
{[\text { resp. } f(\theta \boldsymbol{u}+(1-\theta) \boldsymbol{v}) \leq f(\boldsymbol{v})] .}
\end{array}
$$

Furthermore, if the weak inequality " $\geq$ " (resp. " $\leq$ ") is replaced by the strict inequality " $>$ " (resp. " $<$ "), $f$ is said to be strictly quasiconcave (resp. strictly quasiconvex).

(a)

(b)

(c)

Figure 12.2: The graphic illustrations of quasiconcavity and quasiconvexity: (a) The function is strictly quasiconcave; (b) the function is strictly quasiconvex; and (c) the function is quasiconcave but not strictly quasiconcave.

Remark 12.4.1 From the definition of quasiconcavity (resp. quasiconvexity), we know that quasiconvity (resp. quasiconvexity) is a weaker condition than concavity (resp. convexity).

Theorem I (Negative of a function). If $f(\boldsymbol{x})$ is quasiconcave (resp. strictly quasiconcave), then $-f(\boldsymbol{x})$ is quasiconvex (resp. strictly quasiconvex).

Theorem II (concavity versus quaisconcavity). Any (strictly) concave (resp. convex) function is (strictly) quasiconcave (resp. quasiconvex), but the converse may not be true.

Theorem III (linear function). If $f(x)$ is linear, then it is quasiconcave as well as quasiconvex.

Theorem IV (monotone function with one variable). If $f$ is a function of one variable, then it is quasiconcave as well as quasiconvex.

Remark 12.4.2 Note that, unlike concave (convex) functions, a sum of two quasiconcave (quasiconvex) functions is not necessarily quasiconcave (resp. quasiconvex).

(a)

(b)

(c)

Figure 12.3: The graphic representation of the alternative definitions of quasiconcavity and quasiconvexity.

From Figure 12.3, we can see that a function is quasiconcave (resp. quasiconvex) if and only if the upper contour set $S^{\geq}(k) \equiv\{\boldsymbol{x} \in X: f(\boldsymbol{x}) \geq k\}$ (resp. the lower contour set $S \leq(k) \equiv\{\boldsymbol{x} \in X: f(\boldsymbol{x}) \leq k\}$ ) is convex.

Sometimes it is easier use this alternative (equivalent) definitions to check quasiconcavity and quasiconvexity. We state it here as a proposition.

Proposition 12.4.1 A function $f(\boldsymbol{x})$, where $\boldsymbol{x}$ is a vector of variables in the domain $X$, is quasiconcave (resp. quasiconvex) if and only if, for any constant $k$, the upper contour set $S \geq(k) \equiv\{\boldsymbol{x} \in X: f(\boldsymbol{x}) \geq k\}$ (resp. the lower contour set $S \leq(k) \equiv\{\boldsymbol{x} \in X: f(\boldsymbol{x}) \leq k\})$ is convex.

Proof. Necessity: Suppose that $f$ is quasiconcave. Let $\boldsymbol{x}_{1}$ and $\boldsymbol{x}_{2}$ be two points of $S \geq(k)$. We need to show: all weighted averages $\boldsymbol{x}_{\theta} \equiv$ $\theta \boldsymbol{x}_{1}+(1-\theta) \boldsymbol{x}_{2}, \theta \in[0,1]$ are in $S^{\geq}(k)$.

Since $\boldsymbol{x}_{1} \in S^{\geq}(k)$ and $\boldsymbol{x}_{2} \in S^{\geq}(k)$, by the definition of upper contour set, we have $f\left(\boldsymbol{x}_{1}\right) \geqq k$ and $f\left(\boldsymbol{x}_{2}\right) \geqq k$.

Now, for any $\boldsymbol{x}_{\theta}$, since $f$ is quasi-concave, then:

$$
f\left(\boldsymbol{x}_{\theta}\right) \geqq \min \left[f\left(\boldsymbol{x}_{1}\right), f\left(\boldsymbol{x}_{2}\right)\right] \geqq k .
$$

Therefore, $f\left(\boldsymbol{x}_{\theta}\right) \geqq k$, and then $\boldsymbol{x}_{\theta} \in S^{\geq}(k)$. Consequently, $S S^{\geq}(k)$ must be a convex set.

Sufficiency: we need to show: if for all $k \in \mathbb{R}, S \geq(k)$ is a convex set, then $f(\boldsymbol{x})$ is a quasi-concave function. Let $\boldsymbol{x}_{1}$ and $\boldsymbol{x}_{2}$ be two arbitrary points in $X$. Without loss of generality, suppose $f\left(\boldsymbol{x}_{1}\right) \geqq f\left(\boldsymbol{x}_{2}\right)$. Since for all $k \in \mathbb{R}, S^{\geq}(k)$ is a convex set, then $S^{\geq}\left(f\left(\boldsymbol{x}_{2}\right)\right)$ must be convex. It is also clear that $\boldsymbol{x}_{2} \in S^{\geq}\left(f\left(\boldsymbol{x}_{2}\right)\right)$, and since $f\left(\boldsymbol{x}_{1}\right) \geqq f\left(\boldsymbol{x}_{2}\right)$, we have $\boldsymbol{x}_{1} \in S^{\geq}\left(f\left(\boldsymbol{x}_{2}\right)\right)$. As such, for any weighted average of $\boldsymbol{x}_{1}$ and $\boldsymbol{x}_{2}$, we must have $\boldsymbol{x}_{\theta} \in S^{\geq}\left(f\left(\boldsymbol{x}_{2}\right)\right)$. It follows from the definition of $S^{\geq}\left(f\left(\boldsymbol{x}_{2}\right)\right)$ that $f\left(\boldsymbol{x}_{\theta}\right) \geqq f\left(\boldsymbol{x}_{2}\right)$. As a consequence, we must have

$$
f\left(\boldsymbol{x}_{\theta}\right) \geqq \min \left[f\left(\boldsymbol{x}_{1}\right), f\left(\boldsymbol{x}_{2}\right)\right] .
$$

Therefore, $f(\boldsymbol{x})$ is quasi-concave.

Example 12.4.1 (1) $Z=x^{2}$ is quasiconvex since $S \leq$ is convex.
(2) $Z=f(x, y)=x y$ is quasiconcave since $S^{\geq}$is convex.
(3) $Z=f(x, y)=(x-a)^{2}+(y-b)^{2}$ is quasiconvex since $S \leq$ is convex.
(4) $Z=f(x, y)=\frac{1}{x}+\frac{1}{y}$ is quasiconcave since $S^{\geq}$is convex.

The above facts can be seen by looking at graphs of these functions.

## Differentiable Functions

Similar to the concavity, when a function is differentiable, we have the following proposition.

Proposition 12.4.2 Suppose that $f: \mathbb{R} \rightarrow \mathbb{R}$ is differentiable. Then $f$ is quasiconcave if and only if for any $x, y \in \mathbb{R}$, we have

$$
\begin{equation*}
f(y) \geqq f(x) \Rightarrow f^{\prime}(x)(y-x) \geqq 0 \tag{12.4.8}
\end{equation*}
$$

When there are two or more independent variables, the above proposition becomes:

Proposition 12.4.3 Suppose that $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is differentiable. Then $f$ is quasiconcave if and only if for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{n}$, we have

$$
\begin{equation*}
f(\boldsymbol{y}) \geqq f(\boldsymbol{x}) \Rightarrow \sum_{j=1}^{n} \frac{\partial f(\boldsymbol{x})}{\partial x_{j}}\left(y_{j}-x_{j}\right) \geqq 0 \tag{12.4.9}
\end{equation*}
$$

If a function $z=f\left(x_{1}, x_{2}, \cdots, x_{n}\right)$ is twice continuously differentiable, quasiconcavity and quansiconvexity can be checked by means of the first and second order partial derivatives of the function.

Define a bordered determinant as follows:

$$
|\boldsymbol{B}|=\left|\begin{array}{ccccc}
0 & f_{1} & f_{2} & \cdots & f_{n} \\
f_{1} & f_{11} & f_{12} & \cdots & f_{1 n} \\
f_{2} & f_{21} & f_{22} & \cdots & f_{2 n} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
f_{n} & f_{n 1} & f_{n 2} & \cdots & f_{n n}
\end{array}\right| .
$$

Remark 12.4.3 The determinant $|\boldsymbol{B}|$ is different from the bordered Hessian $|\boldsymbol{H}|$. Unlike $|\boldsymbol{H}|$, the border in $|\boldsymbol{B}|$ is composed of the first and second derivatives of the function $f$ rather than an extraneous constraint function $g$ and Lagrange function $Z$.

We can define the successive principal minors of $\boldsymbol{B}$ as follows:

$$
\left|\boldsymbol{B}_{1}\right|=\left|\begin{array}{cc}
0 & f_{1}, \\
f_{1} & f_{11}
\end{array}\right|, \quad\left|\boldsymbol{B}_{2}\right|=\left|\begin{array}{ccc}
0 & f_{1} & f_{2} \\
f_{1} & f_{11} & f_{12} \\
f_{2} & f_{21} & f_{22}
\end{array}\right|, \cdots, \quad\left|\boldsymbol{B}_{n}\right|=|\boldsymbol{B}| .
$$

A necessary condition for a function $z=f\left(x_{1}, \cdots, x_{n}\right)$ defined the nonnegative orthant to be quasiconcave is that

$$
\left|\boldsymbol{B}_{1}\right| \leq 0, \quad\left|\boldsymbol{B}_{2}\right| \geq 0, \quad\left|\boldsymbol{B}_{3}\right| \leq 0, \quad \cdots, \quad(-1)^{n}\left|\boldsymbol{B}_{n}\right| \geq 0 .
$$

A sufficient condition for $f$ to be strictly quasiconcave on the nonnegative orthant is that

$$
\left|\boldsymbol{B}_{1}\right|<0, \quad\left|\boldsymbol{B}_{2}\right|>0, \quad\left|\boldsymbol{B}_{3}\right|<0, \cdots, \quad(-1)^{n}\left|\boldsymbol{B}_{n}\right|>0 .
$$

For strict quasiconvexity, the corresponding sufficient condition is that

$$
\left|\boldsymbol{B}_{1}\right|<0,\left|\boldsymbol{B}_{2}\right|<0, \cdots, \quad\left|\boldsymbol{B}_{n}\right|<0 .
$$

Example 12.4.2 $z=f\left(x_{1}, x_{2}\right)=x_{1} x_{2}$ for $x_{1}>0$ and $x_{2}>0$. Since $f_{1}=x_{2}$, $f_{2}=x_{1}, f_{11}=f_{22}=0$, and $f_{12}=f_{21}=1$, the relevant principal minors turn out to be

$$
\left|\boldsymbol{B}_{1}\right|=\left|\begin{array}{cc}
0 & x_{2} \\
x_{2} & 0
\end{array}\right|=-x_{2}^{2}<0, \quad\left|\boldsymbol{B}_{2}\right|=\left|\begin{array}{ccc}
0 & x_{2} & x_{1} \\
x_{2} & 0 & 1 \\
x_{1} & 1 & 0
\end{array}\right|=2 x_{1} x_{2}>0 .
$$

Thus $z=x_{1} x_{2}$ is strictly quasiconcave on the positive orthant.

Example 12.4.3 Show that $z=f(x, y)=x^{a} y^{b}(x, y>0 ; a, b>0)$ is quasiconcave.

Since

$$
\begin{aligned}
& f_{x}=a x^{a-1} y^{b}, f_{y}=b x^{a} y^{b-1} \\
& f_{x x}=a(a-1) x^{a-2} y^{b}, f_{x y}=a b x^{a-1} y^{b-1}, f_{y y}=b(b-1) x^{a} y^{b-2}
\end{aligned}
$$

thus

$$
\begin{aligned}
&\left|\boldsymbol{B}_{1}\right|=\left|\begin{array}{cc}
0 & f_{x} \\
f_{x} & f_{x x}
\end{array}\right|=-\left(a x^{a-1} y^{b}\right)^{2}<0 \\
&\left|\boldsymbol{B}_{2}\right|=\left|\begin{array}{ccc}
0 & f_{x} & f_{y} \\
f_{x} & f_{x x} & f_{x y} \\
f_{y} & f_{y x} & f_{y y}
\end{array}\right|=\left[2 a^{2} b^{2}-a(a-1) b^{2}-a^{2} b(b-1)\right] x^{3 a-2} y^{3 b-2} \\
&=a b(a+b) x^{3 a-2} y^{3 b-2}>0
\end{aligned}
$$

Hence it is strictly quasiconcave.

Remark 12.4.4 When the constraint function $g(\boldsymbol{x})$ is linear: $g(\boldsymbol{x})=a_{1} x_{1}+$ $\cdots+a_{n} x_{n}=c$, the second-order partial derivatives of $g$ vanish, and thus, from the first-order condition $f_{i}=\lambda g_{i}$, the bordered determinant $|\boldsymbol{B}|$ and
the bordered Hessian determinant have the following relationship:

$$
|\boldsymbol{B}|=\lambda^{2}|\overline{\boldsymbol{H}}| .
$$

Consequently, in the linear constraint case, the bordered determinant $|\overline{\boldsymbol{B}}|$ and the bordered Hessian $|\overline{\boldsymbol{H}}|$ always have the same sign at the stationary point of $z$. The same holds for the order of the leading principal minors. Therefore, if the bordered determinant $|\overline{\boldsymbol{B}}|$ satisfies the sufficient condition for strict quasiconcavity, the bordered Hessian $|\overline{\boldsymbol{H}}|$ must also satisfy the second-order sufficient condition for constrained maximization.

## Absolute versus Relative Extrema

If an objective function is strictly quasiconcave (quasiconvex), and the constraint function is convex, by similar reasoning for concave (convex) functions, its relative constrained maximum (minimum) is a unique absolute maximum (absolute minimum), following similar reasoning for concave (resp. convex) functions.

Theorem 12.4.1 (Global Optimum) Suppose that the objective function $f$ is twice differentiable and strictly quasiconcave, and the constraint functions $g^{j}(\boldsymbol{x})$ are differentiable and quasiconvex on $X \subseteq \mathbb{R}^{n}$. If $\boldsymbol{x}^{*}$ is a stationary point of $X$, then $f\left(\boldsymbol{x}^{*}\right)$ is a unique global constrained maximum.

### 12.5 Utility Maximization and Consumer Demand

Let us now examine the consumer choice problem - utility maximization problem. For simplicity, only consider the two-commodity case. The consumer wants to maximize her utility:

$$
u=u(x, y) \quad\left(u_{x}>0, u_{y}>0\right)
$$

subject to her budget constraint

$$
P_{x} x+P_{y} y=I
$$

by taking prices $P_{x}$ and $P_{y}$ as well as his income $I$ as given.

## First-Order Condition

The Lagrange function is

$$
Z=u(x, y)+\lambda\left(I-P_{x} x-P_{y} y\right) .
$$

At the first-order condition, we have the following equations:

$$
\begin{aligned}
& Z_{\lambda}=I-P_{x} x-P_{y} y=0 ; \\
& Z_{x}=u_{x}-\lambda P_{x}=0 \\
& Z_{y}=u_{y}-\lambda P_{y}=0 .
\end{aligned}
$$


(a)

(b)

Figure 12.4: The graphical illustration of the conditions for utility maximization.

From the last two equations, we have

$$
\frac{u_{x}}{P_{x}}=\frac{u_{y}}{P_{y}}=\lambda
$$

or

$$
\frac{u_{x}}{u_{y}}=\frac{P_{x}}{P_{y}}
$$

The term $\frac{u_{x}}{u_{y}} \equiv M R S_{x y}$ is the so-called marginal rate of substitution of $x$ for $y$. Thus, we obtain the well-known equality: $M R S_{x y}=\frac{P_{x}}{P_{y}}$ which is the necessary condition for the interior solution.

## Second-Order Condition

If the bordered Hessian in the present problem is positive, i.e., if

$$
|\overline{\boldsymbol{H}}|=\left|\begin{array}{ccc}
0 & P_{x} & P_{y} \\
P_{x} & u_{x x} & u_{x y} \\
P_{y} & u_{y x} & u_{y y}
\end{array}\right|=2 P_{x} P_{y} u_{x y}-P_{y}^{2} u_{x x}-P_{x}^{2} u_{y y}>0
$$

(with all the derivatives evaluated at the stationary point of $\bar{x}$ and $\bar{y}$ ), then the stationary value of $u$ will assuredly be maximum.

Since the budget constraint is linear, from the result in the previous section, we have

$$
|\boldsymbol{B}|=\lambda^{2}|\overline{\boldsymbol{H}}| .
$$

Thus, as long as $|\boldsymbol{B}|>0$, we know the second-order condition holds.
Recall that $|\boldsymbol{B}|>0$ means that the utility function is strictly quasiconcave.

Also, the quasi-concavity of a utility function means that the indifference curves represented by the utility function are convex, i.e., the upper contour set $y: u(y)) \geq u(x)$ is convex, and in this case, we say that the preferences represented by the utility function are convex.

Remark 12.5.1 The convexity of preferences implies that consumers wan$t$ to diversify their consumption, and thus, convexity can be viewed as the formal expression of the basic measure of economic markets for diversification. Also, strict quasi-concavity implies the strict convexity of $\succ_{i}$, which in turn implies the conventional diminishing marginal rates of substitution (DMRS), and weak convexity of $\succcurlyeq_{i}$ is equivalent to the quasiconcavity of the utility function $u_{i}$.

From $M R S_{x y}=\frac{P_{x}}{P_{y}}$, we can solve for $x$ or $y$ as a function of the other and then substitute it into the budget line to find the demand function for $x$ or $y$.

Example 12.5.1 Consider that the Cobb-Douglas utility function:

$$
u(x, y)=x^{a} y^{1-a}, \quad 0<a<1
$$

which is strictly increasing and concave on $\mathbb{R}_{++}^{2}$.
Substituting $M R S_{x y}=\frac{M U_{x}}{M U_{y}}=\frac{a y}{(1-a) x}$ into $M R S_{x y}=\frac{P_{x}}{P_{y}}$, we have

$$
\frac{a y}{(1-a) x}=\frac{P_{x}}{P_{y}}
$$

and then

$$
y=\frac{(1-a) P_{x} x}{a P_{y}} .
$$

Substituting the above $y$ into the budget line $P_{x} x+P_{y} y=I$ and solving for $x$, we get the demand function for $x$

$$
x\left(P_{x}, P_{y}, I\right)=\frac{a I}{P_{x}} .
$$

Substituting the above $x\left(P_{x}, P_{y}, I\right)$ into the budget line, the demand

220 CHAPTER 12. OPTIMIZATION WITH EQUALITY CONSTRAINTS
function for $y$ is obtained:

$$
y\left(P_{x}, P_{y}, I\right)=\frac{(1-a) I}{P_{y}}
$$

## Chapter 13

## Optimization with Inequality

## Constraints

Classical optimization methods, such as the method of Lagrange multipliers, handle optimization problems with equality constraints in the form of $g\left(x_{1}, \ldots, x_{n}\right)=c$. On the other hand, non-classical optimization, also known as mathematical programming, deals with problems that have inequality constraints, such as $g\left(x_{1}, \ldots, x_{n}\right) \leq c$.

Mathematical programming encompasses linear programming and nonlinear programming. In linear programming, the objective function and all inequality constraints are linear. If either the objective function or an inequality constraint is nonlinear, the problem is one of nonlinear programming.

In the following, we restrict our attention to non-linear programming.

### 13.1 Non-Linear Programming

The nonlinear programming problem is that of choosing nonnegative values of certain variables so as to maximize or minimize a given (non-linear)
function subject to a given set of (non-linear) inequality constraints.
The nonlinear programming maximum problem is

$$
\begin{array}{ll} 
& \max \quad f\left(x_{1}, \ldots, x_{n}\right) \\
\text { s.t. } & g^{i}\left(x_{1}, \ldots, x_{n}\right) \leq b_{i}, \quad i=1,2, \ldots, m ; \\
& x_{1} \geq 0, \ldots, x_{n} \geq 0
\end{array}
$$

Similarly, the minimization problem is

$$
\begin{array}{ll} 
& \min \quad f\left(x_{1}, \ldots, x_{n}\right) \\
\text { s.t. } & g^{i}\left(x_{1}, \ldots, x_{n}\right) \geq b_{i}, \quad i=1,2, \ldots, m ; \\
& x_{1} \geq 0, \ldots, x_{n} \geq 0 .
\end{array}
$$

Firstly, it is worth noting that unlike the case of equality constraints, there are no restrictions on the relative size of $m$ and $n$ in nonlinear programming problems. Secondly, the direction of the inequalities ( $\leq$ or $\geq$ ) in the constraints is only a convention since an inequality like $g^{i} \leq b_{i}$ can be easily converted to the opposite inequality by multiplying both sides by -1 , yielding $-g^{i} \geq-b_{i}$. Thirdly, an equality constraint, say $g^{k}=b_{k}$, can be replaced by two inequality constraints, $g^{k} \leq b_{k}$ and $-g^{k} \leq-b_{k}$.

Now, let us define the term "binding constraint":

Definition 13.1.1 (Binding Constraint) A constraint $g^{j} \leq b_{j}$ is called binding (or active) at $\boldsymbol{x}^{0}=\left(x_{1}^{0}, \ldots, x_{n}^{0}\right)$ if $g^{j}\left(x^{0}\right)=b_{j}$, i.e., if $\boldsymbol{x}^{0}$ is a boundary point of the constraint.

### 13.2 Kuhn-Tucker Conditions

For the purpose of eliminating certain irregularities on the boundary of the feasible set, a restriction is imposed on the constrained functions. This restriction is known as the constraint qualification. The following is a stronger version of the constraint qualification, which is easier to verify.

Definition 13.2.1 Let $C$ be the constraint set. We say that the constraint qualification condition is satisfied at $\boldsymbol{x}^{*} \in C$ if the gradients (vectors of partial derivatives) of the $g^{j}$-constraints associated with all binding constraints at $\boldsymbol{x}^{*}$ are linearly independent for $j=1, \ldots, m$.

We define the Lagrangian function for optimization with inequality constraints as:

$$
L\left(x_{1}, \ldots, x_{n}, \lambda_{1}, \ldots, \lambda_{m}\right)=f\left(x_{1}, \ldots, x_{n}\right)+\sum_{j=1}^{m} \lambda_{j}\left(b_{j}-g^{j}\left(x_{1}, \ldots, x_{n}\right)\right) .
$$

The following result is the necessity theorem for Kuhn-Tucker conditions for a local optimum.

Theorem 13.2.1 (Necessity Theorem of Kuhn-Tucker Conditions) Suppose the objective functions and constraint functions are differentiable, and the constraint qualification condition is satisfied. Then, we have:
(1) the Kuhn-Tucker necessary condition for maximization is:

$$
\begin{aligned}
& \frac{\partial L}{\partial x_{i}} \leq 0, \quad x_{i} \geq 0 \quad \text { and } \quad x_{i} \frac{\partial L}{\partial x_{i}}=0, i=1, \ldots, n \\
& \frac{\partial L}{\partial \lambda_{j}} \geq 0, \quad \lambda_{j} \geq 0 \quad \text { and } \quad \lambda_{j} \frac{\partial L}{\partial \lambda_{j}}=0, j=1, \ldots, m
\end{aligned}
$$

(2) the Kuhn-Tucker necessary condition for minimization is:

$$
\frac{\partial L}{\partial x_{i}} \geq 0, \quad x_{i} \geq 0 \quad \text { and } \quad x_{i} \frac{\partial L}{\partial x_{i}}=0, i=1, \ldots, n
$$

$$
\frac{\partial L}{\partial \lambda_{j}} \leq 0, \quad \lambda_{j} \geq 0 \quad \text { and } \quad \lambda_{j} \frac{\partial L}{\partial \lambda_{j}}=0, j=1, \ldots, m
$$

Note that, from $x_{i} \frac{\partial L}{\partial x_{i}}=0$, as long as $x_{i}>0$, we must have

$$
\frac{\partial L}{\partial x_{i}}=0 .
$$

Similarly, as long as $\lambda_{j}>0$, we have

$$
\frac{\partial L}{\partial \lambda_{j}}=0 .
$$

Therefore, the first-order condition can be equivalently expressed as:
(1) the Kuhn-Tucker necessary condition for maximization is:

$$
\begin{aligned}
\frac{\partial L}{\partial x_{i}} & \leq 0, \quad \text { with equality if } x_{i}>0, i=1, \ldots, n \\
\frac{\partial L}{\partial \lambda_{j}} & \geq 0, \quad \text { with equality if } \lambda_{j}>0, j=1, \ldots, m
\end{aligned}
$$

(2) the Kuhn-Tucker necessary condition for minimization is:

$$
\begin{aligned}
& \frac{\partial L}{\partial x_{i}} \geq 0, \quad \text { with equality if } x_{i}>0, i=1, \ldots, n \\
& \frac{\partial L}{\partial \lambda_{j}} \leq 0, \quad \text { with equality if } \lambda_{j}>0, j=1, \ldots, m
\end{aligned}
$$

Example 13.2.1 Consider the following nonlinear program:

$$
\max \quad \pi=x_{1}\left(10-x_{1}\right)+x_{2}\left(20-x_{2}\right)
$$

$$
\begin{array}{ll}
\text { s.t. } & 5 x_{1}+3 x_{2} \leq 40 \\
& x_{1} \leq 5 \\
& x_{2} \leq 10 \\
& x_{1} \geq 0, x_{2} \geq 0 .
\end{array}
$$

The Lagrangian function of the nonlinear program in Example (13.2.1) is:
$L=x_{1}\left(10-x_{1}\right)+x_{2}\left(20-x_{2}\right)-\lambda_{1}\left(5 x_{1}+3 x_{2}-40\right)-\lambda_{2}\left(x_{1}-5\right)-\lambda_{3}\left(x_{2}-10\right)$.

The Kuhn-Tucker conditions are:

$$
\begin{gathered}
\frac{\partial L}{\partial x_{1}}=10-2 x_{1}-5 \lambda_{1}-\lambda_{2} \leq 0 \\
\frac{\partial L}{\partial x_{2}}=20-2 x_{2}-3 \lambda_{1}-\lambda_{2} \leq 0 \\
\frac{\partial L}{\partial \lambda_{1}}=-\left(5 x_{1}+3 x_{2}-40\right) \geq 0 ; \\
\frac{\partial L}{\partial \lambda_{2}}=-\left(x_{1}-5\right) \geq 0 \\
\frac{\partial L}{\partial \lambda_{3}}=-\left(x_{2}-10\right) \geq 0 \\
x_{1} \geq 0, x_{2} \geq 0 \\
\lambda_{1} \geq 0, \lambda_{2} \geq 0, \lambda_{3} \geq 0 \\
x_{1} \frac{\partial L}{\partial x_{1}}=0, \quad x_{2} \frac{\partial L}{\partial x_{2}}=0 \\
\lambda_{i} \frac{\partial L}{\partial \lambda_{i}}=0, \quad i=1,2,3 .
\end{gathered}
$$

The constraint qualification for the nonlinear program in Example (13.2.1) is satisfied since all constraints are linear and functionally independent . Therefore, the optimal solution $\left(\frac{95}{34}, \frac{295}{34}\right)$ must satisfy the Kuhn-Tucker
condition.

When comparing the Kuhn-Tucker Theorem with Lagrange multipliers in the context of equality-constrained optimization problems, the major difference lies in the signs of the multipliers. Specifically, Kuhn-Tucker multipliers are nonnegative, while Lagrange multipliers can be positive or negative. This additional information can be useful in various scenarios.

The Kuhn-Tucker Theorem only provides a necessary condition for a maximum. The following theorem states conditions that guarantee the sufficiency of the above first-order conditions.

Theorem 13.2.2 (Kuhn-Tucker Sufficiency Theorem) Suppose that the following conditions are satisfied:
(a) $f$ is differentiable and satisfies the condition:

$$
\begin{equation*}
D f(\boldsymbol{x})\left(\boldsymbol{x}^{\prime}-\boldsymbol{x}\right)>0 \text { for any } \boldsymbol{x} \text { and } \boldsymbol{x}^{\prime} \text { with } f\left(\boldsymbol{x}^{\prime}\right)>f(\boldsymbol{x}), \tag{13.2.1}
\end{equation*}
$$

which is satisfied if one of the following two conditions is satisfied:
(a.i) $f$ is concave.
(a.ii) $f$ is quasi-concave and $\operatorname{Df}(\boldsymbol{x}) \neq 0$ for all $x \in \mathbb{R}_{+}^{n}$.
(b) Each constraint function is differentiable and quasi-convex.
(c) $\boldsymbol{x}^{*}$ satisfies the Kuhn-Tucker condition and Constraint qualification condition is satisfied at $\boldsymbol{x}^{*}$.

Then $\boldsymbol{x}^{*}$ is a global maximizer.

Proof. Suppose $\boldsymbol{x}^{*}$ is not a global maximizer. Then, $f\left(\boldsymbol{x}^{\prime}\right)>f\left(\boldsymbol{x}^{*}\right)$ for some $\boldsymbol{x}^{\prime}$ satisfying $g_{i}\left(\boldsymbol{x}^{\prime}\right) \leq d_{i}$ for every $i$. Then, by condition (13.2.1), we have $D f\left(\boldsymbol{x}^{*}\right)\left(\boldsymbol{x}^{\prime}-\boldsymbol{x}^{*}\right)>0$. If $\lambda_{i}>0$, the Kuhn-Tucker condition implies
$g_{i}\left(\boldsymbol{x}^{*}\right)=d_{i}$. Moreover, since $g_{i}(\cdot)$ is quasi-convex and $g_{i}\left(\boldsymbol{x}^{\prime}\right) \leq d_{i}=g_{i}\left(\boldsymbol{x}^{*}\right)$, we have $D g_{k}\left(\boldsymbol{x}^{*}\right)\left(\boldsymbol{x}^{\prime}-\boldsymbol{x}^{*}\right) \leq 0$. Therefore, we have both $D f\left(\boldsymbol{x}^{*}\right)\left(\boldsymbol{x}^{\prime}-\boldsymbol{x}^{*}\right)>0$ and $\sum_{i} \lambda_{k} D g_{k}\left(\boldsymbol{x}^{*}\right)\left(\boldsymbol{x}^{\prime}-\boldsymbol{x}^{*}\right) \leq 0$, contradicting the Kuhn-Tucker condition since it requires that $D f\left(\boldsymbol{x}^{*}\right)=\sum_{i} \lambda_{k} D g_{k}\left(\boldsymbol{x}^{*}\right)\left(\boldsymbol{x}^{*}\right)$.

If instead we have $D f(\boldsymbol{x})\left(\boldsymbol{x}^{\prime}-\boldsymbol{x}\right)<0$ for all $\boldsymbol{x}$ and $\boldsymbol{x}^{\prime}$ with $f(\boldsymbol{x})>$ $f\left(\boldsymbol{x}^{\prime}\right)$ and the multipliers have the nonpositive sign that corresponds to a minimization problem, then $\boldsymbol{x}^{*}$ is a global minimizer.

In particular, when there is only one constraint, let $C=\left\{\boldsymbol{x} \in \mathbb{R}^{n}\right.$ : $g(\boldsymbol{x}) \leq d\}$. We have the following proposition.

Proposition 13.2.1 Suppose $f$ is quasi-concave with $D f(\boldsymbol{x}) \neq 0$ for all $x \in \mathbb{R}_{+}^{n}$, and $C$ is a convex set (if $g$ is quasi-convex, then this result holds). If $\boldsymbol{x}$ satisfies the Kuhn-Tucker first-order conditions, then $\boldsymbol{x}$ is a global solution to the constrained maximization problem.

The problem of finding the nonnegative vector $\left(x^{*}, \lambda^{*}\right), x^{*}=\left(x_{1}^{*}, \ldots, x_{n}^{*}\right)$, $\lambda^{*}=\left(\lambda_{1}^{*}, \ldots, \lambda_{m}^{*}\right)$, which satisfies the Kuhn-Tucker necessary condition and for which
$L\left(x, \lambda^{*}\right) \leq L\left(x^{*}, \lambda^{*}\right) \leq L\left(x^{*}, \lambda\right) \quad \forall \quad x=\left(x_{1}, \ldots, x_{n}\right) \geq 0, \lambda=\left(\lambda_{1}, \ldots, \lambda_{m}\right) \geq 0$
is known as the saddle point problem.

Proposition 13.2.2 If $\left(x^{*}, \lambda^{*}\right)$ solves the saddle point problem, then $\left(x^{*}, \lambda^{*}\right)$ solves the problem (13.1.1).

### 13.3 Economic Applications

## Corner Solution for Linear Utility Maximization

Suppose the preference ordering is represented by the linear utility function :

$$
u(x, y)=a x+b y
$$

Since the marginal rate of substitution of $x$ for $y$ is $a / b$ and the economic rate of substitution of $x$ for $y$ is $p_{x} / p_{y}$, they cannot be equal in general, as long as $\frac{a}{b} \neq \frac{p_{x}}{p_{y}}$. In this case, the solution to the utility maximization problem typically involves a boundary solution: only one of the two goods will be consumed.

To find the optimal consumption, we can use the Kuhn-Tucker theorem, which is the appropriate tool to use here, since we will almost never have an interior solution.

The Lagrange function is

$$
L(x, y, \lambda)=a x+b y+\lambda\left(m-p_{x} x-p_{y} y\right)
$$

and thus

$$
\begin{align*}
& \frac{\partial L}{\partial x}=a-\lambda p_{x}  \tag{13.3.2}\\
& \frac{\partial L}{\partial y}=b-\lambda p_{t}  \tag{13.3.3}\\
& \frac{\partial L}{\partial \lambda}=m-p_{x}-p_{y} . \tag{13.3.4}
\end{align*}
$$

There are four cases to be considered:

1. $x>0$ and $y>0$. Then we have $\frac{\partial L}{\partial x}=0$ and $\frac{\partial L}{\partial y}=0$. Thus, $\frac{a}{b}=\frac{p_{x}}{p_{y}}$. Since $\lambda=\frac{a}{p_{x}}>0$, we have $p_{x} x+p_{y} y=m$ and thus all $x$ and $y$ that satisfy $p_{x} x+p_{y} y=m$ are the optimal consumption.
2. $x>0$ and $y=0$. Then we have $\frac{\partial L}{\partial x}=0$ and $\frac{\partial L}{\partial y} \leq 0$. Thus, $\frac{a}{b} \geq \frac{p_{x}}{p_{y}}$. Since $\lambda=\frac{a}{p_{x}}>0$, we have $p_{x} x+p_{y} y=m$ and thus $x=\frac{m}{p_{x}}$ is the optimal consumption.
3. $x=0$ and $y>0$. Then we have $\frac{\partial L}{\partial x} \leq 0$ and $\frac{\partial L}{\partial y}=0$. Thus, $\frac{a}{b} \leq \frac{p_{x}}{p_{y}}$. Since $\lambda=\frac{b}{p_{y}}>0$, we have $p_{x} x+p_{y} y=m$ and thus $y=\frac{m}{p_{y}}$ is the optimal consumption.
4. $x=0$ and $y=0$. Since the utility function is strictly increasing, we have $p_{x} x+p_{y} y=m$. However, since $m \neq 0$, this case is impossible.

In summary, the demand functions are given by

$$
\left(x\left(p_{x}, p_{y}, m\right), y\left(p_{x}, p_{y}, m\right)\right)= \begin{cases}\left(m / p_{x}, 0\right) & \text { if } a / b>p_{x} / p_{y} \\ \left(0, m / p_{y}\right) & \text { if } a / b<p_{x} / p_{y} \\ \left(x, m / p_{x}-p_{y} / p_{x} x\right) & \text { if } a / b=p_{x} / p_{y}\end{cases}
$$

for all $x \in\left[0, m / p_{x}\right]$.


Figure 13.1: Utility maximization for linear utility function

Remark 13.3.1 In fact, it is easily found out the optimal solutions by comparing relatives steepness of the indifference curves and the budget line. For instance, consider Figure 13.1 below. If $\frac{a}{b}>\frac{p_{x}}{p_{y}}$, the indifference curves become steeper as we move downwards and to the right, so the optimal solution is the one where the consumer spends all their income on good
$x$. On the other hand, if $\frac{a}{b}<\frac{p_{x}}{p_{y}}$, the indifference curves become flatter as we move downwards and to the right, so the optimal solution is the one where the consumer spends all their income on good $y$. If $\frac{a}{b}=\frac{p_{x}}{p_{y}}$, the indifference curves and the budget line are parallel and coincide at the optimal solutions, and thus the optimal solutions are given by all the points on the budget line.

## Economic Interpretation of the Kuhn-Tucker Condition

A maximization program in the general form, for example, is the production problem facing a firm which has to produce $n$ goods such that it maximizes its revenue subject to $m$ resource (factor) constraints.

The variables have the following economic interpretations:

- $x_{j}$ is the amount produced of the $j t h$ product;
- $r_{i}$ is the amount of the $i t h$ resource available;
- $f$ is the profit (revenue) function;
- $g^{i}$ is a function which shows how the $i$ th resource is used in producing the $n$ goods.

The optimal solution to the maximization program indicates the optimal quantities of each good the firm should produce.

In order to interpret the Kuhn-Tucker condition, we first have to note the meanings of the following variables:

- $f_{j}=\frac{\partial f}{\partial x_{j}}$ is the marginal profit (revenue) of product $j$;
- $\lambda_{i}$ is the shadow price of resource $i$;
- $g_{j}^{i}=\frac{\partial g^{i}}{\partial x_{j}}$ is the amount of resource $i$ used in producing a marginal unit of product $j$;
- $\lambda_{i} g_{j}^{i}$ is the imputed cost of resource $i$ incurred in the production of a marginal unit of product $j$.

The condition $\frac{\partial L}{\partial x_{j}} \leq 0$ can be written as $f_{j} \leq \sum_{i=1}^{m} \lambda_{i} g_{j}^{i}$ and it says that the marginal profit of the $j$ th product cannot exceed the aggregate marginal imputed cost of the $j$ th product.

The Kuhn-Tucker condition $x_{j} \frac{\partial L}{\partial x_{j}}=0$ implies that, in order to produce good $j\left(x_{j}>0\right)$, the marginal profit of good $j$ must be equal to the aggregate marginal imputed $\operatorname{cost}\left(\frac{\partial L}{\partial x_{j}}=0\right)$. The same condition shows that good j is not produced $\left(x_{j}=0\right)$ if there is an excess imputation $x_{j} \frac{\partial L}{\partial x_{j}}<0$.

The Kuhn-Tucker condition $\frac{\partial L}{\partial \lambda_{i}} \geq 0$ is simply a restatement of constraint $i$, which states that the total amount of resource $i$ used in producing all the $n$ goods should not exceed total amount available $r_{i}$.

The condition $\frac{\partial L}{\partial \lambda_{i}}=0$ indicates that if a resource is not fully used in the optimal solution $\left(\frac{\partial L}{\partial \lambda_{i}}>0\right)$, then its shadow price will be $0\left(\lambda_{i}=0\right)$. On the other hand, a fully used resource $\left(\frac{\partial L}{\partial \lambda_{i}}=0\right)$ has a strictly positive price $\left(\lambda_{i}>0\right)$.

Example 13.3.1 Let us find an economic interpretation for the maximization program given in Example (13.2.1). Consider a firm that produces two goods using three types of resources available in limited quantities: 40 units of the first resource, 5 units of the second resource, and 10 units of the third resource. The first resource is used in the production of both goods, where 5 units are required to produce one unit of good 1, and 3 units are required to produce one unit of good 2 . The second resource is only used to produce good 1, while the third resource is only used to produce good 2.

The prices of the two goods are given by the linear inverse demand equations $p_{1}=10-x_{1}$ and $p_{2}=20-x_{2}$, where $x_{1}$ and $x_{2}$ are the quantities of goods 1 and 2 produced, respectively. The firm's objective is to maxi-
mize its revenue $R=x_{1} p_{1}+x_{2} p_{2}$ subject to the resource constraints and the non-negativity constraints $x_{1} \geq 0$ and $x_{2} \geq 0$.

The optimization problem can be formulated as:

$$
\begin{aligned}
\max & R=x_{1}\left(10-x_{1}\right)+x_{2}\left(20-x_{2}\right) \\
\text { s.t. } & 5 x_{1}+3 x_{2} \leq 40 ; \\
& x_{1} \leq 5 ; \\
& x_{2} \leq 10 ; \\
& x_{1} \geq 0, x_{2} \geq 0
\end{aligned}
$$

which is exactly the maximization program proposed in Example (13.2.1) The optimal solution to this problem is $\left(x_{1}, x_{2}\right)=(2,10)$, which indicates that the firm should produce 2 units of good 1 and 10 units of good 2 in order to maximize its revenue.


[^0]:    ${ }^{1}$ These lecture notes are prepared for the purpose of my teaching and the convenience of my students during class. Kindly refrain from distributing them.

